Paulo Vargas Moniz

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Quantum Cosmology - The Supersymmetric Perspective - Vol. 1

Fundamentals

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To all *my cats!*

Preface

We read in order to know we are not alone, I once heard, and perhaps it could also be suggested that *we write in order not to be alone*, to endorse, to promote continuity.

The idea for this book took about ten years to materialize, and it is the author's hope that its content will constitute the beginning of further explorations beyond current horizons. More specifically, this book appeals to the reader to engage upon and persevere with a journey, moving through the less well explored territories in the evolution of the very early universe, and pushing towards new landscapes. Perhaps, during or after consulting this book, this attitude and this willingness will be embraced by someone, somewhere, and this person will go on to enrich our quantum cosmological description of the early universe, by means of a clearer supersymmetric perspective. It is to these creative and inquisitive 'young minds' that the book is addressed.

The reader will not therefore find in this book all the answers to all the problems regarding a supersymmetric and quantum description of the early universe, and this remark is substantiated in the book by a list of unresolved and challenging problems, itself incomplete.

Consequently, the idea is to provide a description of the many features present in a supersymmetric perspective of quantum cosmology. The book is split into two volumes:

- In Vol. I, entitled *Fundamentals*, the reader will find an accessible primer. After a *contextualized introduction and guidance* through possible routes of exploration, the essential content starts with a chapter presenting quantum cosmology in general terms. It is then followed by a chapter summarizing the ideas and methods of supersymmetry and supergravity. It is only afterwards that a thorough supersymmetric analysis of some relevant (quantum) cosmological models is undertaken. More precisely, the reader and fellow explorer is introduced to the main ideas, techniques, achievements, and problems, which characterize the research subject of *supersymmetric quantum cosmology* (SQC). Different approaches that have been employed in SQC will be discussed, bringing them together for the first time in a book publication.
- In Vol. II, entitled *Advanced Topics*, the scope for analyzing quantum cosmological models within a supersymmetric framework is broadened. The aim is

to provide those who have worked through Vol. I with an introduction to *more* topics, as well as some complementary developments. Volume II adds furthers essentials and additional (optional) frameworks to employ within SQC.

The above objectives for this book will be helped along with a detailed set of exercises, accompanied by the corresponding solutions. A display of summary boxes (outlining relevant features, concepts, and results in the form of reviewing questions) will be added at the end of each chapter.

It is hoped that this book will stimulate the interest of (a) final year undergraduates, (b) graduate students, (c) lecturers designing a course that will include aspects of superstrings and supergravity from a quantum mechanical *and* cosmological point of view, and (d) researchers who would like to either initiate or apply SQC methods and ideas in their work. The reader is assumed to have a good working knowledge of mathematical analysis, quantum mechanics, modern cosmology, field theory, and general relativity theory. A prior knowledge of quantum cosmology is *not* a condition to start using this graduate textbook. In particular, concepts related to supersymmetry/supergravity or quantum cosmology will generally be presented within the book as indicated.

Finally, before embarking upon the many technical details of this fascinating exploration, I would like to share with the reader the following words by Alberto Caeiro (Fernando Pessoa):

Aceita o Universo
Como to deram os deuses.
Se os deuses te quisessem dar outro
ter-to-iam dado.
Se há outras matérias e outros mundos –
Haja.

London, Köln, and Covilhã Paulo Vargas Moniz

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Most of my learning on superstring, supergravity, quantum cosmology, and then supersymmetric quantum cosmology was obtained at DAMTP, University of Cambridge. I spent five wonderful years there (1993–1998), learning a lot and getting better at it all the time. I would particularly like to thank G. Gibbons, P. Townsend, and M. Perry for useful discussions and hints. Special words go to S.W. Hawking and P.D. D'Eath for their time, patience, and sharing. In particular, I must thank P.D. D'Eath for making available his vast learning with regard to supersymmetric quantum cosmology. A.D.Y. Cheng was both a friend and a stimulating collaborator,

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Finally, a few words are owed to the following persons, although no words could ever be sufficient. First to my dear mother and father, I hope this book will honour *all* the time, patience, dedication, effort, and teaching you unconditionally gave to me, providing me with the guidance to reach this far. And, of course, the author is deeply and immensely grateful to his dear wife, Teresa, for her love, patience, and support at *all* times and beyond what any words could express, but also to his dear daughter, Teresa, for keeping him awake regularly at the crack of dawn.

So, now, on with the book.

London, Köln, and Covilhã, Paulo Vargas Moniz
November 2004–December 2008

Contents

Chapter 1
Introduction

What is this book about? What is quantum cosmology with supersymmetry? How is supersymmetry implemented? Is it through the use of (recent developments in) a superstring theory? Why should the very early universe be explored in that manner? Are there enticing and interesting research problems left to solve? How relevant would it be to address and solve them?

The above are just the kind of questions that a potential reader is likely to ask as she or he begins to browse through the first few pages of this book. Perhaps a clear picture will only emerge by the time the final chapters are reached. But still, we may say in simplified terms that investigating quantum cosmology with supersymmetry means using both bosonic and fermionic degrees of freedom, supersymmetrically intertwined, within a suitable quantum mechanical representation of the universe (e.g., where bosonic canonical momenta are represented by a differential operator and fermions by a matrix).

Supersymmetry (SUSY) [1–7] will be implemented through different procedures, ranging from taking the bosonic sector of different string theories and then inserting fermionic partners in an appropriately consistent manner, up to using the full theory of $N = 1$ supergravity (SUGRA) [8–10] in 4D spacetime. Of course, this may seem to fall short of using the whole set of elements of 10-dimensional superstring or 11-dimensional SUGRA: $N = 1$ 11-dimensional SUGRA is considered to be a low energy limit of M-theory, constituting each of the five 10-dimensional weak-coupling limits of superstring theories (all these limits are related by duality transformations) [11–13]. However, when initiating and proceeding to explore uncharted domains, it is often more efficient to get acquainted with the main features of simpler settings before attempting bolder routes,[1] hoping the results found in the former will be in some way employed in the latter.

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[1] $N = 1$ 4D SUGRA can follow from 11D or 10D SUGRA through suitable compactifications and restrictions. Other 4D limits are, e.g., $N = 2$ SUGRA (which constitutes *the* sought route to bring Einstein's general relativity and Maxwell's electromagnetism within a unified setting), and $N = 8$ SUGRA [8] (which was long considered the best hope for a unified theory of all interactions and quantum gravity).

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Superstring theory (and its extensions to SUGRA or M-theory) [14–19, 11–13], although still under development, is an excellent (perhaps the best!) candidate for a *unification* theory, constituting a fascinating quantum gravity framework [20]. The limits of investigation have been stretched impressively, in particular, allowing the proposal of novel scenarios that go deep into the very early universe. It is therefore hoped that superstring theory will indeed assist in the process of stepping into these and many other as yet unexplored landscapes (see, e.g., [21, 22]).

Starting with superstring theory (and requiring the extra spatial dimensions to be compactified) to achieve a 4D spacetime description, it is found that the resulting quantum gravity scenario could retain at least some supersymmetry. Such a setting would surely be interesting for the cosmology of the very early universe. It could, on the one hand, assist in reducing our ignorance regarding the nature of the creation of the universe, and on the other hand, hopefully provide *lateral* information that would enrich our mastery of superstring theory, perhaps contributing indirectly to clarifying its central ideas and its complete formulation.

To be more precise, exploring the very early universe with the help of features from a quantum gravity theory will hopefully bring more accurate explanations, elucidating observational issues and other perplexing problems in contemporary cosmology. For example, what is the inflaton? What is the nature of the dark energy effect? Why do we have this universe and not another? Moreover, the analysis of the very early universe (ranging from a quantum origin up to structure formation, and involving a crucial inflationary stage) currently also offers a noteworthy opportunity to test some of the features and predictions of such a quantum gravity theory. In fact, cosmology has quite recently reached a significant level of observational accuracy, prompting part of the scientific community to identify this stage as a 'golden epoch' [23–26]. Quantum gravity or its superstring version may eventually acquire an observational component in the context of future cosmological tests, quite apart from speculative hopes for the LHC in 2009 and beyond.

Consequently, a quantum mechanical approach to cosmological theories retrieved via a quantum gravity theory may constitute a significant step in the study of the early evolution of the universe. This methodology is generally called quantum cosmology (QC). It aims to explain *how* and *why* our universe is the way it is. Basically, quantum cosmology employs quantum mechanics to investigate the universe as a whole [27–36].

Research on quantum cosmology has been through several periods. In the 1960s, the seminal work of C. Misner, J. Wheeler, and B. DeWitt built the foundations of canonical quantum gravity [37–45]. In particular, they established the conditions (constraint equations) that the quantum state of the universe ought to satisfy, together with a definition of its configuration space (superspace). In this context, the universe is studied by means of a wave function, rather than classical spacetime solutions. Almost all the models subsequently considered had all but a finite number of degrees of freedom *frozen*. This is achieved by restricting the fields to be spatially homogeneous, and such models correspond to a (finite-dimensional) 'minisuperspace' scenario [29, 30, 46, 33, 47, 35, 36]. It was only in the 1980s that quantum cosmology attracted renewed and active interest. The main reason was the rigorous debate and introduction of boundary conditions for the wave function of the universe. J. Hartle and S.W. Hawking, on the one hand, [28, 48], and A. Vilenkin [49–51], on the other, put forward the two main schemes for boundary conditions, with suitable variations added by A. Linde. The vast majority of research has been divided among what are known as the no-boundary proposal and the tunneling proposal, although recent work in superstring quantum cosmology and the *landscape problem* has made use of the so-called infinite wall proposal advanced earlier by B. DeWitt (see, e.g., [42]).

The physical setting in which the overwhelmingly vast majority of publications in quantum cosmology can be found is that of Einstein's general relativity within a Hamiltonian formulation or a Feynman path integral, employing a metric point of view [28, 30, 48, 46, 20]. But, as mentioned above, if we wish to proceed within a more fundamental perspective, elements from superstring/supergravity theory would constitute an attractive and perhaps more fundamental angle from which to explore the very early universe and explain how the universe got its observational properties. For these reasons, the use of such supersymmetric frameworks for viewing quantum cosmology could not be ignored, and have indeed been explored over the past 25 years or so [52–56].

This setting, in which the methods of quantum cosmology are extended to embrace the techniques of SUSY, is called *supersymmetric quantum cosmology* (SQC) [52, 57]. SQC thus constitutes an interesting and rewarding research topic. On the one hand, it provides the opportunity to perform calculations that may be relevant for phenomenology, and on the other hand, it is closely linked to exciting new areas of fundamental research, such as superstring theory (and theoretical high energy physics in general). In brief, the fundamental purpose of SQC research (in the author's opinion) is to determine whether a path can be consistently established from the description of a supersymmetric quantum universe (built upon some of the *intrinsic* elements of superstring and supergravity theories) toward the current view of a classical stage [58, 59, 56], possibly with observationally testable predictions.

Let us add that the presence of SUSY has constituted an element of the utmost importance in quantum gravity investigations. For example, it plays a crucial role in SUGRA and superstring theory in removing divergences [8] or unstable states (e.g., tachyons) that would otherwise be present in bosonic quantum gravity theories (e.g., simply extending from general relativity or the bosonic string case) [17–19, 13].

It must also be emphasized that supergravity theories represent a kind of *square root* of Einsteinian gravity. To determine physical states it may be sufficient to employ the Lorentz and supersymmetry invariances [60, 61]. Moreover, it has been pointed out [62] that the presence in purely bosonic quantum cosmological models of string dualities can be related to the existence of quantum states that have invariance under SUSY.

In more detail, the following properties and features further enhance a significant motivation for SQC:

- Both pure quantum gravity and SUSY effects are treated as equally determinant, guaranteeing an improved description of the very early universe [52, 57, 59, 56]. This contrasts with bosonic quantum cosmology, where quantum gravity is present but supersymmetry is *not*. In the SQC framework, we will therefore find a larger set of variables (bosonic and fermionic), as well as additional symmetries which increase the number of constraints, subsequently imposing a wider algebra.
- $N = 1$ SUGRA [63, 8] is employed as a natural square root of gravity in the manner of Dirac [55, 64, 60, 61]. The analysis of a second order equation of the Klein–Gordon type (i.e., the Wheeler–DeWitt equation) could be replaced by the analysis of a supersymmetrically induced set of first order differential equations. This would then have profound consequences regarding *what* to take as a (SUSY) wave function of the universe and *how* to retrieve it [58, 65–68] and any corresponding cosmological behaviour subsequently induced.
- There remain many open issues in SQC [57]. A brief list is as follows:
  - Does SQC improve on the purely bosonic quantum cosmological formulations with matter fields but no SUSY? That is, does the presence of SUSY invariance contribute, and is it paramount to a more realistic (quantum mechanical) description of the very early universe?
  - How can spontaneous supersymmetry breaking be *properly* described within a SQC perspective [59]?
  - How can we analyse the retrieval of semiclassical features and the origin of structure formation in SQC?
  - Is it possible to consistently identify quantum-to-classical transitions in SQC?
  - Is there an imprint of an early SUSY quantum epoch on the (currently) observed universe [67]?

The key purpose of this book will be to introduce and present the essentials, as well as pertinent details of SQC to the interested fellow explorer, in order to address some of the above challenges. The book is split into two volumes, each divided into four parts.

Volume I conveys, in essence, the *fundamentals* of SQC. Part I constitutes a brief overview of the methods and results discussed throughout the two volumes and is provided identically in both books. It contains this introduction as Chap. 1.

Part II of Vol. I presents some basic elements that will be required for the core part of the book. Chapter 2 provides a generic description of the main aspects of and methods used in quantum cosmology, namely, minisuperspace quantization. We introduce and explain the basic principles of SUSY and SUGRA in Chap. 3. The Hamiltonian formulation and canonical quantization of SUGRA are analysed in detail in Chap. 4.

In Part III, Chap. 5 focuses on supersymmetric cosmological models extracted from $N = 1$ SUGRA, exploring a fermionic differential operator representation for SQC models, which has proved itself to be a suitable method. The emphasis is on FRW models, and in particular on obtaining physical states for quantum supersymmetric universes. Bianchi models are also dealt with, and the presence of matter fields is explored. Chapter 6 presents a thorough description of supersymmetric minisuperspaces derived from the bosonic sectors of superstring theories. Some of these models are explored in a context where duality properties can be related to these *hidden* supersymmetries.

Finally, in Part IV, within the context of Vol. I, we will present an appraisal of current accomplishments, and point to subsequent suggestions for investigation in SQC. In more detail, Chap. 7 will describe the main results that have been achieved, while Chap. 8 lists several lines of further enquiry.

Two appendices conclude the volume, explaining the notation used along with some useful expressions (Appendix A), and describing canonical quantization procedures for theories with constraints (Appendix B).

Volume II is entitled *Advanced Topics*. It begins with a carbon copy of Part I from Vol. I, the intention being to provide a consistent context and broad guidance throughout the book. Part II is entitled *Further Essentials*. These begin in Chap. 2, which corresponds to Chap. 2 of Vol. I, focusing again on quantum cosmology and possible semiclassical limits. Chapter 2 provides a brief treatment of SUSY breaking in SUSY and SUGRA theories, placing particular emphasis on supersymmetric quantum mechanics (SQM). Then in Chap. 4, counterpart to Chap. 4 of Vol. I, we detail a framework suitable for describing the semiclassical limit of quantum SUGRA.

Part III of Vol. II discusses alternative frameworks for SQC. The presence of supermatter and SUSY breaking is presented in Chap. 5 with regard to supersymmetric cosmological models extracted from $N = 1$ SUGRA, with a fermionic differential operator representation for SQC. Then in Chap. 6 we describe connection and loop variables in SQC, with emphasis on the interesting developments they have stimulated. Chapter 7 deals with the matrix fermionic representation used in SQC for fermionic momenta. Chapter 8 reports on minisuperspace models extending from those discussed in Chap. 6 of Vol. I.

We conclude with Part IV. Chapter 9 summarizes reported accomplishments concerning the frameworks introduced in Parts II and Part III of Vol. II, while Chap. 10 suggests more routes for exploration and investigation that follow on from the contents of Vol. II. An appendix concludes Vol. II, with a review of the notation and some further useful expressions.

Having said all this, it is time to wish the reader a pleasant and stimulating journey of exploration within the current confines of supersymmetric quantum cosmology (SQC) and of course beyond, if she or he should so desire. Because there are indeed plenty of challenging and open problems still to address, whose resolution may change our knowledge and perspective of *how* a realistic very early universe might have come into being and subsequently evolved.

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Chapter 2
A Survey of Quantum Cosmology

Why consider a research subject like quantum cosmology (QC)? How did it become relevant, to the point of having a vast number of conferences (or parallel sessions), books, and published articles devoted to it? There follows a sequence of possible arguments in favour of QC.

2.1 Motivation and Justification

Contemporary cosmology is a well-established quantitative area, where remarkable new technology has been used to build up a precise chart of the universe [1, 2]. In particular, fundamental cosmological parameters have been displayed recently with outstanding precision. At the dawn of the twenty-first century, the cosmology community has thus entered a *golden epoch*, where future improvements (in both quantity and quality) will lead to an even clearer perspective of *where* we are, and *why* and *how* we come to be here [3–5].

The current paradigm in cosmology is the *inflationary 'big bang'* scenario [6–16]. It has been under scrutiny, but so far has successfully passed all the major tests. And it has allowed us to address some of the observational inconsistencies of the standard cosmological model:

- Nearby regions currently observed and spatially separated would not have been (as determined by the standard cosmological dynamics) in thermal contact, and yet the isotropy of the cosmic microwave background radiation (CMBR) forces us to consider otherwise. This is the *horizon* problem.
- For the currently observed spatial flatness, the universe had to be flat at early times to an incredible accuracy, in fact, to within 10^{-50}. This is the *flatness* problem.
- One expects topological defects to have been produced in phase transitions in the early stages of evolution, but they are not observed.

These problems can be explained by an early 'inflationary' phase of exponential-like expansion of the universe. Causal contact then becomes possible in the primeval past, at the same time as the universe enlarges so much that locally it becomes

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nearly flat, with topological defects becoming effectively unobservable. Inflation also provides a suitable mechanism for initial small *quantum* matter perturbations to increase and form a fluctuation spectrum, which is consistent with observations.

However, an apparent weakness (see also [17–19]) emerges for this picture. For this paradigm to be realistic [10, 14], it has to be generically possible. We need to know the probability for the inflationary scenario to occur. Moreover, we need to know *how* those perturbations arose. The problem is that, notwithstanding its many merits, these questions lie beyond the scope of the inflationary paradigm.

In fact, this is the issue of the *initial conditions* of the universe (see detailed discussions in [20–38]). Let us first indicate briefly *why* the discussion of initial conditions is crucial. We can then focus on *ways* to investigate their generality.

Consider therefore a universe whose geometry is described by a (Lemaître)–Friedmann–Robertson–Walker metric (abbreviated in the following to LFRW or FRW) of the form

$$\mathrm{d}s^2 = \varsigma^2 \left[-\mathcal{N}^2(t)\mathrm{d}t^2 + \mathrm{e}^{2\alpha(t)}\mathrm{d}\Omega_3^2(k) \right] , \tag{2.1}$$

where $\varsigma^2 \equiv 2/(3\pi M_{\mathrm{P}}^2)$, $\mathrm{d}\Omega_3^2(k)$ denotes the metric of the spatial sections with constant spatial curvature labelled by $k = 0, \pm 1$, and $a(t) \equiv \mathrm{e}^{\alpha(t)}$ is the scale factor (see Sects. A.1 and A.2). For the matter content, a time-dependent scalar field $\sqrt{2}\pi\varsigma\phi(t)$, with a potential $2\pi^2\varsigma^2 V(\phi)$, can be included, and the action is [20]

$$S = \frac{1}{2}\int \mathrm{d}t\mathcal{N}\mathrm{d}\alpha \left[-\frac{\dot{\alpha}^2}{\mathcal{N}^2} + \frac{\dot{\phi}^2}{\mathcal{N}^2} - V(\phi) + k\mathrm{e}^{-2\alpha} \right] , \tag{2.2}$$

where dots over functions indicate differentiation with respect to (proper) time. The classical equations of motion are

$$-\frac{a\dot{a}^2}{\mathcal{N}^2} + \frac{a^3\dot{\phi}^2}{\mathcal{N}^2} - ka + a^3 V = 0 , \tag{2.3}$$

$$\frac{1}{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{\mathcal{N}}\right) + \frac{2a\dot{\phi}^2}{\mathcal{N}^2} - aV = 0 , \tag{2.4}$$

$$\frac{1}{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\phi}}{\mathcal{N}}\right) + 3\frac{\dot{a}\dot{\phi}}{a\mathcal{N}^2} + \frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 . \tag{2.5}$$

Here $\mathcal{N}$, called the *lapse* function, is explained in the next section. It is not of physical relevance classically.[1] $V(\phi)$ is generic, and within some range it will be large, satisfying the *slow-roll* approximation (see, e.g., [10])

[1] An alternative (proper) time parameter can always be chosen (as happens in theories where there is time reparametrizatrion invariance) through $\mathrm{d}\tau = \mathcal{N}\mathrm{d}t$, including the gauge $\mathcal{N} = 1$, whence t is then the proper time.

$$\left| \frac{1}{V(\phi)} \frac{\mathrm{d}V}{\mathrm{d}\phi} \right| \ll 1 \,.$$

At this point the attentive reader will point out that a suitable domain for an inflationary phase is easily identified, namely, $\dot{\phi} \sim 0$, where the potential acts as an effective cosmological constant, leading to $a(t) \sim \mathrm{e}^{\sqrt{V}t}$. However, and this is the *crucial* issue, that evolution requires a *choice* of initial conditions. The theory of dynamical systems applied to the above equations leads to a two-parameter family of physically distinct solutions, i.e., trajectories in the corresponding phase space. The case of the $k = +1$ solutions is particularly pertinent as it illustrates the dependence on initial conditions. In fact, if initial values of ϕ and $\dot{\phi}$ are restricted to start away from the $k = 0$ curve with large $|\dot{\phi}|$, the universe recollapses, without a sufficient inflationary phase, while otherwise inflation occurs. So the choice of the initial values of ϕ and $\dot{\phi}$ is indeed crucial for the occurrence of satisfactory inflation to be a general feature.

But classically speaking we have no guide as to how to choose one set of initial conditions rather than another. Additional arguments are therefore required. One option is to invoke *quantum* cosmological ingredients, e.g., the universe began in some sort of transition from a quantum regime, so that the initial classical parameters are determined in a probabilistic way. However, as our resilient explorer may also have concluded, the task seems then to transfer our quest into determining (a) the most probable[2] state (wave function) of the universe, and (b) its distinctive predictive signatures.

Before proceeding with such issues, it must also be said that QC is basically the application of quantum mechanics to models with time reparametrization invariance (e.g., general relativity). Indeed, QC can be considered as a kind of toy model attempt to obtain the relevant information for a full quantum theory of gravity [39–42]. This must surely be one of the supreme challenges for fundamental science in the twenty-first century.

Note 2.1 On the one hand, general relativity is not perturbatively renormalisable. Efforts by at least one generation of physicists to import the features of quantum physics into it have not yet been successful. On the other hand, general relativity is an appropriate theory for dealing with the larger scales of spacetime, while quantum physics applies instead to extremely small scales. Furthermore, the concept of quantum gravity also involves quantizing spacetime itself, and not merely quantizing the matter fields present in that spacetime background. In spite of all these (apparently) serious obstacles, the problem cannot be avoided. Overwhelming observational data seems to indicate that the universe did start out very, very small indeed (the cosmological singularity), and quantum mechanics, as the essential framework for dealing with such small scales, ought to be applied to this very early universe.

[2] Assuming that the probability interpretation still holds.

Note 2.2 In the 1930s, L. Rosenfeld and M. Bronstein [41] were perhaps the first to construct a quantum linearized theory, whereas the canonical approach was introduced quite some time afterwards by P. Bergmann (1956) [43]. Then in the 1950s, P. Dirac [44–46] and Pirani and Schild (1950) [43] began to probe the intrinsic non-linearities. The famous trio Arnowitt, Deser, and Misner (ADM) were active from the 1960s, [47–49] when they constructed a Hamiltonian theory of gravitation, whose quantization was soon followed in the work of DeWitt [50–55, 40, 56] and Wheeler [57–59], with significant contributions by Ryan in the 1970s [43] and Misner [60, 49, 61–63]. But it was only toward the mid-1980s that a new surge of interest appeared, thanks to the fundamental contributions of Hawking, Hartle, and Vilenkin, but also Linde, Halliwell, Kiefer, and many, many others who began to consider initial conditions and predictions from the wave function [20, 21, 25–28, 64, 65, 31–37].

2.2 Hamiltonian Formulation of General Relativity

From the previous section, it is clear that, in order to advance towards a QC point of view, the essentials of a quantum formulation of gravity have to be analysed. This has usually embraced a corresponding Hamiltonian description of general relativity, where the ADM [47–49, 43] (or $3 + 1$) decomposition of spacetime is mandatory.[3] At this point the following thought may emerge. Quantizing gravity straight from a Hamiltonian formulation of general relativity is surely not the only option. In fact, the approaches adopted in SUGRA (see Chap. 3) and superstring theories [66, 67] have brought additional elements and perspectives. In the following, we will employ 4D general relativity via a metric description.

2.2.1 *The ADM (or 3 + 1) Decomposition*

Making a $3 + 1$ split of the 4D spacetime manifold $\mathcal{M}$ basically means foliating it into spatial hypersurfaces Σ_t, labeled by a *global* time parameter t. The spacetime dimensional metric [with a $(- + ++)$ Lorentzian signature] is[4]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\omega^0 \otimes \omega^0 + h_{ij}\omega^i \otimes \omega^j \,, \tag{2.6}$$

[3] Quantum geometrodynamics (based on quantum states dependent on the spatial 3-metrics) is perhaps the most widely known [52, 60, 59], but quantum connection dynamics (i.e., using non-Abelian connections) [39, 42] and more specifically holonomies [39, 42] (leading to loop quantum gravity), have recently attracted much interest.

[4] For a definition of units used herein, see Sects. A.1 and A.2.

where we use the basis

$$\omega^0 \equiv \mathcal{N} \mathrm{d}t \,, \qquad \omega^i \equiv \mathrm{d}x^i + \mathcal{N}^i \mathrm{d}t \,. \tag{2.7}$$

A few remarks are in order (see Fig. 2.1):

- This decomposition requires the manifold $\mathcal{M}$ to be *globally hyperbolic*.
- $\mathcal{N}(t, x^k)$ is called the *lapse* function. It measures the difference between the *coordinate* time t and the *proper* time τ along curves normal to the hypersurfaces Σ_t. The normal n is $n_\mu = (-\mathcal{N}, 0, 0, 0)$.
- The quantity $\mathcal{N}^i(t, x^k)$ is the *shift* vector. It measures the difference[5] between a spatial point P and the point one would reach if, instead of following P from one hypersurface to the next, one followed a curve tangent to the normal n.
- ${}^{(3)}h_{ij}(t, x^k) \equiv h_{ij}(t, x^k)$ is the *intrinsic* 3-metric (also called the *first fundamental form*), induced on the spatial hypersurfaces by the full 4D metric $g_{\mu\nu}$.
- In components (matrix representation),

$$g_{\mu\nu} = \begin{bmatrix} -\mathcal{N}^2 + \mathcal{N}_i \mathcal{N}^i & \mathcal{N}_j \\ \mathcal{N}_i & h_{ij} \end{bmatrix} . \tag{2.8}$$

The next step requires the definition of a specific curvature quantity. Besides the usual *intrinsic* curvature tensor ${}^{(3)}R^i{}_{jkl}(h_{mn})$ describing the 3D spatial curvature intrinsic to the hypersurfaces Σ_t, retrieved solely from the intrinsic metric h_{mn},

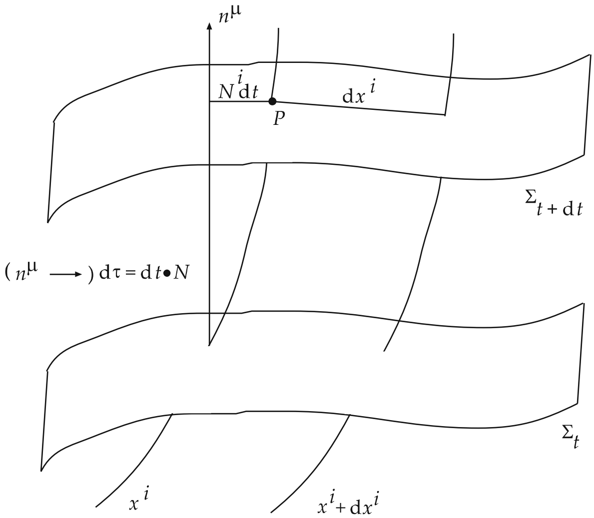

Fig. 2.1 ADM slicing of the spacetime manifold, in terms of the lapse function $\mathcal{N}$ and the shift vector $\mathcal{N}^i$, with surfaces labelled by a time parameter t

[5] If $\mathcal{N}^i = 0$, the spatial coordinates are *comoving*.

the *extrinsic* curvature K_{ij} or *second fundamental form* describes how the spatial hypersurfaces Σ_t become curved regarding the physical spacetime manifold within which they are embedded. This can be computed[6] from $n_{i;j}$ according to

$$K_{ij} \equiv -n_{i;j} = \Gamma^0{}_{ij} n_0 = -\mathcal{N}\Gamma^0{}_{ji} = \frac{1}{2\mathcal{N}}\left(\mathcal{N}_{i|j} + \mathcal{N}_{j|i} - \frac{\partial h_{ij}}{\partial t}\right), \tag{2.9}$$

where the semi-colon indicates spacetime covariant differentiation with respect to the metric $g_{\mu\nu}$, and the vertical bar denotes covariant differentiation with respect to the spatial metric[7] h_{ij}. With these remarks, let us now see how to obtain a Hamiltonian structure.

2.2.2 Hamiltonian and Constraints

For general relativity, the gravitational dynamics is determined by the Einstein–Hilbert action (with a possible cosmological term Λ, and a matter sector):

$$S = \frac{1}{2\mathsf{k}^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left({}^4R - 2\Lambda\right) + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{\mathrm{matter}}\,, \tag{2.10}$$

where $\mathsf{k}^2 \equiv 8\pi G = 8\pi M_{\mathrm{Planck}}^{-2}$, $K \equiv K^i{}_i$ is the trace of the extrinsic curvature, $g \equiv \det g_{\mu\nu}$, and $h \equiv \det h_{ij}$. Many possibilities for S_{matter} have been used.[8] In particular, (2.10) is relevant for probing the generality of inflation from a quantum mechanical origin and therefore the matter has usually been taken in the form of a scalar field ϕ, with potential $V(\phi)$ [11, 12], leading to

$$S_{\mathrm{matter}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{2.11}$$

> *Note 2.3* The term $2\int_{\partial M} \mathrm{d}^3x\sqrt{h}K$ in (2.10) is referred to as a *boundary term* [41].

At this point, the action (2.10) with (2.11) must be brought into the framework of the $3+1$ ADM split. Two important elements are

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[6] As a coordinate basis, we take $\mathbf{e}_i$ tangent to a spacelike hypersurface and the normal $\mathbf{n}$ with components $\mathbf{n} \to n_\mu = (-\mathcal{N}, 0, 0, 0)$.

[7] $\mathcal{N}_{i|j} \equiv \mathcal{N}_{i,j} - \Gamma^k{}_{ij}\mathcal{N}_k$.

[8] A rather incomplete list could be as follows: Abelian vectors [20, 68, 43], non-Abelian fields [69, 70], strings [66, 67], fermions [71], fermions and SUSY [72].

$$^{(4)}R = {}^{(3)}R - K_{ij}K^{ij} + K^2 - 2\,{}^{(4)}R_{\perp}{}^{\alpha}{}_{\perp\alpha} \tag{2.12}$$

and

$$^{(4)}R_{\perp}{}^{\alpha}{}_{\perp\alpha} = \left(n^{\gamma} n^{\beta}{}_{;\alpha} - n^{\beta} n^{\gamma}{}_{;\gamma}\right)_{;\beta} - K^{ij}K_{ij} + K^2\,, \tag{2.13}$$

where

$$K = -n^{\gamma}{}_{;\gamma}\,, \qquad K^{ij}K_{ji} = n^{\beta}{}_{;\gamma} n^{\gamma}{}_{;\beta}\,, \tag{2.14}$$

and we are using the notation

$$-A_{\perp} = A^{\perp} \equiv -n^{\mu}A_{\mu} = \mathcal{N}A^0 = -\frac{1}{\mathcal{N}}\left(A_0 - \mathcal{N}^i A_i\right)\,, \tag{2.15}$$

which will become clearer in Sect. 2.5. Hence, the action becomes [20, 41, 38]

$$S \equiv \int \mathrm{d}t\, L = \frac{1}{2\mathsf{k}^2}\int \mathrm{d}t\mathrm{d}^3x\mathcal{N}\sqrt{h}\left[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\right] + S_{\mathrm{matter}}\,. \tag{2.16}$$

With a scalar field ϕ for matter sector, the canonical momenta are

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{2\mathsf{k}^2}\left[K^{(ij)} - h^{ij}K\right]\,, \tag{2.17}$$

$$\pi_{\phi} \equiv \frac{\delta L}{\delta \dot{\phi}} = \frac{\sqrt{h}}{\mathcal{N}}\left(\dot{\phi} - \mathcal{N}^i \phi_{,i}\right)\,, \tag{2.18}$$

$$\pi^0 \equiv \frac{\delta L}{\delta \dot{\mathcal{N}}} = 0\,, \tag{2.19}$$

$$\pi^i \equiv \frac{\delta L}{\delta \dot{\mathcal{N}}_i} = 0\,. \tag{2.20}$$

The context of (2.19) and (2.20) means that they constitute *primary* constraints (see Appendix B). Alternatively, we can write

$$S = \int \mathrm{d}t\,\mathrm{d}^3x\left(\pi^{ij}\dot{h}_{ij} + \pi_{\phi}\dot{\phi} - \mathcal{N}\mathcal{H}_{\perp} - \mathcal{N}^i\mathcal{H}_i\right)\,, \tag{2.21}$$

or instead, the Hamiltonian

$$\begin{aligned} H &\equiv \int \mathrm{d}^3x\left(\pi^0\dot{\mathcal{N}} + \pi^i\dot{\mathcal{N}}_i + \pi^{ij}\dot{h}_{ij} + \pi_{\phi}\dot{\phi}\right) - L \\ &= \int \mathrm{d}^3x\left(\pi^0\dot{\mathcal{N}} + \pi^i\dot{\mathcal{N}}_i + \mathcal{N}\mathcal{H}_{\perp} + \mathcal{N}_i\mathcal{H}^i\right)\,, \end{aligned} \tag{2.22}$$

with

$$\mathcal{H}_\perp \equiv 2\mathsf{k}^2 \mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{2\mathsf{k}^2}\left[{}^{(3)}R - 2\Lambda\right] + \mathcal{H}_\perp^{\mathrm{matter}}$$
$$= \frac{2\mathsf{k}^2}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{h}\,{}^{(3)}R}{2\mathsf{k}^2} + \frac{1}{2}\sqrt{h}\left(\frac{\pi_\phi^2}{h} + h^{ij}\phi_{,i}\phi_{,j} + 2V\right), \quad (2.23)$$

$$\mathcal{H}^i \equiv -2\pi^{ij}{}_{|j} + h^{ij}\phi_{,j}\pi_\phi\,, \quad (2.24)$$

where

$$\mathcal{G}_{ijkl} = \frac{1}{2}h^{-1/2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right) \quad (2.25)$$

is the *DeWitt metric*. For a wider physical context, see Sect. 2.5. Note in particular that the DeWitt metric allows one to write[9] [52]

$$h^{-1/2}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) = \mathcal{G}_{ijkl}\pi^{ij}\pi^{kl}\,. \quad (2.26)$$

Hence, with the assistance of the ADM 3 + 1 decomposition, the Einstein–Hilbert action has been expressed in a canonical form, from which the field equations can be derived. The fundamental feature is[10] that the Hamiltonian becomes a sum of constraints (see Sect. 2.5 and following note).

Note 2.4 Variation of (2.21) with respect to the lapse function $\mathcal{N}$ produces the *Hamiltonian constraint* $\mathcal{H}_\perp = 0$, while variation of (2.21) with respect to the shift vector $\mathcal{N}_i$ produces the *momentum constraint* $\mathcal{H}^i = 0$. In particular, the lapse and shift functions constitute Lagrange multipliers. These constraints are classified as *secondary* [73, 74] (see also Appendix B). In fact, they constitute a set of constraints on h_{ij}, ϕ, and π^{ij}, π_ϕ, or more precisely, they are the (00) and (0i) components of Einstein's equations. The latter will also correspond to two sets, one for $\partial h_{ij}/\partial t$ and the other for $\partial\pi^{ij}/\partial t$ [52–54, 75, 49, 63].

Although it apparently selects a time coordinate t, the action is invariant under changes in all four coordinates, and is geometrically general at this point.The

[9] An earlier version [43] appeared for a Hamilton–Jacobi formulation by A. Peres. See also [49].

[10] This constitutes the *first* step in the ADM procedure. To be more specific, the ADM method also requires the action of general relativity to be written in the first-order formalism, i.e., with $g_{\mu\nu}$ and $\Gamma^{\mu}_{\nu\alpha}$ as *independent*, followed by the parametrization in (2.8).

essential new feature of (2.21) is the conversion to the variables h_{ij}, ϕ, and π^{ij}, π_ϕ, which represent the degrees of freedom of the system.

Note 2.5 The canonical form of general relativity (and other basic theories of gravity) employs the metric as basic field variable. However, this is already somewhat restrictive. If one has fermionic matter (and therefore spinors, see Chaps. 3 and 4), it is mandatory to use tetrads (or *vierbeine*, the plural of *vierbein*) e^a_μ. Why? A somewhat longer explanation would help, and this can be found in Sects. A.3 and 3.4.3. However, for the moment, we may add somewhat loosely that spinors (fermions) can only be properly defined in the tangent space of the spacetime (manifold), associated with the transformation properties of the Lorentz group and *not* those of a general coordinate transformation group. Hence the need for Lorentz indices or a tetrad structure to project onto the tangent space. The reader is thus invited to attempt Exercise 2.1 on how to express the Einstein–Hilbert action in a canonical form using tetrad fields as basic variables [76–78]. This will prove particularly useful when venturing into SUGRA and its canonical quantization in Chap. 4. For one thing, it brings the *Lorentz generators* into the algebra of constraints of the theory in a natural manner (see Sect. 2.5).

Let us focus on some of the pertinent features of QC, so that later we may identify some routes for exploration of SQC. To be more precise, from (2.21), two approaches can be established for a quantum theory of general relativity. One follows the perspective indicated by Dirac [52, 46, 59], and the other is the ADM procedure [63, 43], which we will focus on in the next section:

- The Dirac approach uses the fact that Einstein's equations can be obtained by variation of the Hamiltonian $\mathcal{N}\mathcal{H}_\perp + \mathcal{N}_i\mathcal{H}^i$. The argument is that, at each point of space, we have only two geometrical dynamical degrees of freedom, viz., h_{ij} and π^{ij}, something the $3+1$ decomposition is telling us. The remaining degrees of freedom will be eliminated from the solutions to the variational equations via the constraints (and other coordinate conditions). In addition, a choice of time (i.e., a choice of gauge) is made by specifying the form of $\mathcal{N}$.
- The ADM procedure requires us to identify $Q^k(h_{ij}, \pi^{ij})$ as *new* coordinates, something that will become clearer in the next section, and to solve the constraints for the corresponding momenta P_k conjugate to Q^k. A *new* Hamiltonian H_{ADM} is established as $-P_0 = H_{\mathrm{ADM}}$, where P_0 is the momentum conjugate to a (possibly new) variable t_{ADM} playing the role of time. The equations of motion of the dynamical degrees of freedom are obtained from H_{ADM}. The time variable is a function $t_{\mathrm{ADM}}(h_{ij}, \pi^{ij})$, and $P_{t_{\mathrm{ADM}}} = -H_{\mathrm{ADM}}$ is solved from $\mathcal{H}_\perp = 0$.

2.3 The ADM Hamiltonian

The contents of this section has not been mainstream QC in recent times, but it is of relevance. On the one hand, its framework is adopted throughout Chap. 7 of Vol. II, and on the other, it is important to acquire a broad view of developments and lines of exploration in QC.

Up to this point, the essence of the ADM procedure has not been implemented. Basically, the action (2.21) is to be further converted (and additionally reduced) to a canonical form. Three steps are required:

- First, solve the constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}^i = 0$ for *four*[11] of the h_{ij}, π^{ij}, and insert the solutions into the action (2.21).
- Second, choose *four* coordinate conditions that will reduce the number of independent variables to just *four*, i.e., aim at *two* of the h_{ij} and *two* of the π^{ij}.
- The action is[12] then reduced to the form [see (2.37) and (2.38) below for the homogeneous case],

$$S \sim \int \left[\pi_{ij}\dot{h}_{ij} - H_{\mathrm{ADM}}(\pi_{ij},\, h_{ij})\right] \mathrm{d}t\mathrm{d}^3x\,. \tag{2.27}$$

These steps may seem somewhat imprecise and open to less objective directions. For example, how do we choose which variables to eliminate, and then what conditions to impose? The answer is that it will depend on the spacetime model and what will be more useful in computational or physical terms. In fact, some choices are dictated by the form the action (2.21) has prior to the above steps (e.g., displaying specific symmetries), and the form it acquires afterwards. Other choices use instead specific restrictions on $\mathcal{N}$ and $\mathcal{N}^i$ to impose some coordinate conditions [which are equivalent in the final result, as long as the variation of (2.21) delivers Einstein's equations].[13] In addition, it may prove useful to solve the $\mathcal{H}^i = 0$ constraint *after* all the above steps have been implemented [43].

2.3.1 Reduction to Homogeneous Cosmologies

Homogeneous cosmologies are particularly amenable to the ADM procedure, and rather interesting results have been produced. For one thing, since the spatial

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[11] There can be up to 12, depending on the symmetries of the spacetime manifold.

[12] The Dirac approach exhibits several significant differences at this point. Although the action is reduced to a similar form, the evolution of the spacetime [in terms of the pair $(\pi_{ij},\, h_{ij})$] is provided through $\mathcal{N}\mathcal{H}_\perp + \mathcal{N}_i\mathcal{H}^i$, which constitutes constraints, along with any other constraints, that are to be imposed with the Hamilton equations.

[13] However, care must be exercised. One can overdetermine the procedure by choosing coordinates *and* choosing, e.g., $\mathcal{N} = 1$, $\mathcal{N}_i = 0$. However, in the end, the equations for $\partial h_{ij}/\partial t$ and $\partial \pi^{ij}/\partial t$ ought to be Einstein's equations.

sections are homogeneous, we can obtain a possible time variable t_{ADM} as the volume $\int \sqrt{h} \mathrm{d}^3 x$, provided it is a monotonic function of t. More precisely, t_{ADM} will be an intrinsic time as a function of the *remaining* h_{ij} and π^{ij}. On the other hand, the final form of the action would have a minimal number of time-dependent variables. For *some* cases within the ADM framework, one or two physical parameters will be enough to characterize the metric, with the cosmological problem becoming that of the motion of a particle in a space of one or two dimensions.[14]

Let us therefore be more precise and consider Bianchi models *without* any matter at this stage. They will be characterized by the (spatial) metric $h_{ij}(t)$ at any instant of time, with the number of degrees of freedom being given by its *six* independent components. It is useful to adopt the Misner–Ryan parametrization [60, 62, 43] in the form[15]

$$h_{ij}(t) = \zeta \mathrm{e}^{-2\Omega(t)} \mathrm{e}^{2\beta}{}_{ij} , \tag{2.28}$$

where $\Omega \sim \ln a$ is a scalar (which will replace the scale factor behavior of a FRW case), ζ is a constant (used when choosing units), and β_{ij} is a traceless symmetric 3×3 matrix. Following Misner, we adopt the coordinate choice

$$t \to \Omega \equiv -\frac{1}{6} \ln \left[\det(h_{ij})\right] , \tag{2.29}$$

making $t_{\text{ADM}} \equiv \Omega$ the *new* time coordinate.

Note 2.6 In Chap. 7 of Vol. II, the formulation of SUGRA employs this Ω time (see (2.28) and (2.29)). In Chaps. 4 and 5 of Vol. I, we use coordinate time t instead, and then in Sect. 6.1, we use an Ω time again, i.e., adopting either the ADM or Dirac procedure for the canonical quantization of $N = 1$ SUGRA, and hence producing *different* models in SQC. In fact, there are differences regarding the quantum states retrieved from the ADM and Dirac procedures even in QC, and it is worth getting acquainted with them before venturing into the SQC domain. In other words, Sects. 2.3 and 2.4 are not merely an exercise for padding out the account here. The purpose is specifically to motivate and prepare subsequent chapters.

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[14] If some constraints remain unsolved, the dimension of the space can go from three to five [43]. If the symmetry allows one to exhaust *all* the degrees of freedom – as happens in the FRW case – quantization will become surprisingly difficult.

[15] Historically speaking, C. Misner worked with (2.28), but in subsequent work, other authors preferred $\Omega \to -\Omega$, i.e., $h_{ij}(t) \sim \mathrm{e}^{2\Omega(t)} \mathrm{e}^{2\beta}{}_{ij} \equiv \mathrm{e}^{2\alpha(t)} \mathrm{e}^{2\beta}{}_{ij}$, with $\alpha \equiv \ln a$, the 'usual' FRW scale factor (see (2.1)). See also [79, 80, 49]. We opt here for (2.28) throughout Sect. 2.3, although the use of $-\Omega$ is also correct. In fact we will maintain $h_{ij}(t) \sim \mathrm{e}^{2\Omega(t)} \mathrm{e}^{2\beta}{}_{ij}$ for Chap. 6 of this volume.

As a consequence, the Hamiltonian can be written with variables β_{ij} as functions of Ω. The next step in the ADM approach is then to determine the *five*[16] independent components β_{ij} as functions of Ω, with the universe generically represented by a point moving through a space of two to five dimensions (associated with the Hamiltonian established using the ADM procedure), under the influence of a potential with *moving* walls [43]. For Bianchi models, we can write

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + h_{ij}(t)\omega^i\omega^j \,, \tag{2.30}$$

where ω^i are one-forms obeying $\mathrm{d}\omega^i = C^i_{jk}\omega^j \wedge \omega^k$ (all nine possible Bianchi types are characterized through the structure constants C^i_{jk} [81, 82]).

For the ADM procedure, the metric must be further expanded, using the *generic* ADM-type decomposition, as follows:

$$\mathrm{d}s^2 = -\left[\mathcal{N}^2(\Omega) + \mathcal{N}^i(\Omega)\mathcal{N}_i(\Omega)\right]\mathrm{d}\Omega^2 + 2\mathcal{N}_i(\Omega)\mathrm{d}\Omega\omega^i + \zeta \mathrm{e}^{-2\Omega}\mathrm{e}^{2\beta(\Omega)}{}_{ij}\omega^i\omega^j \,. \tag{2.31}$$

After integrating over the space variables, the action (2.21) becomes

$$S \sim 2\pi \int \left(\mathrm{e}^{\beta}{}_{is}\pi^s{}_t\mathrm{e}^{-\beta}{}_{tj}\mathrm{d}\beta_{ij} - \pi^k{}_k\mathrm{d}\Omega\right) \,, \tag{2.32}$$

where

$$\mathrm{d}\beta_{ij} \equiv \frac{1}{2}\left[\left(\mathrm{d}\mathrm{e}^{\beta}{}_{is}\right)\mathrm{e}^{-\beta}{}_{sj} + \left(\mathrm{d}\mathrm{e}^{-\beta}{}_{is}\right)\mathrm{e}^{\beta}{}_{sj}\right] \,.$$

Next, we use the definition

$$\mathsf{p}_{ij} \equiv 2\pi\left(\mathrm{e}^{\beta}{}_{is}\pi^s{}_t\mathrm{e}^{-\beta}{}_{tj} - \frac{1}{3}\delta_{ij}\pi^{\ell}{}_{\ell}\right) \,. \tag{2.33}$$

Why do (2.32) and (2.33) have this form? In essence, we wish to write the action in the usual canonical form, where $\mathrm{e}^{\beta}{}_{is}\pi^s{}_t\mathrm{e}^{-\beta}{}_{tj}\mathrm{d}\beta_{ij}$ will be cast as $\mathsf{p}_{ij}\mathrm{d}\beta_{ij}$. Let us therefore write the matrix[17] β in the form

$$\beta \equiv \mathrm{e}^{-\vartheta\kappa_3}\mathrm{e}^{-\theta\kappa_1}\mathrm{e}^{-\varphi\kappa_3}\beta_{\mathrm{d}}\mathrm{e}^{\varphi\kappa_3}\mathrm{e}^{\theta\kappa_1}\mathrm{e}^{\vartheta\kappa_3} \,, \tag{2.34}$$

where κ_1 and κ_3 are matrices, with

$$\beta_{\mathrm{d}} \equiv \mathrm{diag}\left[\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+\right] \,, \tag{2.35}$$

---

[16] Note that in this description a diagonal case requires only *two* components for β_{ij}.

[17] Any real symmetric matrix A with non-zero determinant can be written as $A = R^{-1}A_{\mathrm{d}}R$, where A_{d} is a real diagonal matrix, and R is a rotation matrix such as one of the Euler matrices.

whereupon the action takes the canonical form[18]

$$S \sim \int \left(\mathsf{p}_+ \mathrm{d}\beta_+ + \mathsf{p}_- \mathrm{d}\beta_- + \mathsf{p}_\theta \mathrm{d}\theta + \mathsf{p}_\varphi \mathrm{d}\varphi + \mathsf{p}_\vartheta \mathrm{d}\vartheta - H_{\mathrm{ADM}} \mathrm{d}\Omega \right), \tag{2.36}$$

or in a more compact manner,

$$S \sim \int \left[\mathsf{p}^\varepsilon \mathrm{d}\beta_\varepsilon - H_{\mathrm{ADM}}(\mathsf{p}^\varepsilon, \beta_\varepsilon, \Omega) \mathrm{d}\Omega \right], \tag{2.37}$$

where the Hamiltonian H_{ADM} was obtained by solving $\mathcal{H}_\perp = 0$ (see the steps required in the ADM process indicated above) and is given by[19]

$$H^2_{\mathrm{ADM}} = (2\pi)^2 \left(\pi^\ell{}_\ell \right)^2 = 6\mathsf{p}^{ij}\mathsf{p}_{ij} - 24\pi^2 h\, {}^{(3)}R\,, \tag{2.38}$$

with $\mathsf{p}^{ij}\mathsf{p}_{ij} \equiv \mathrm{Tr}(\mathsf{p})^2$ a quadratic form involving the variables β_+,β_-, θ, φ, ϑ, and $h^{(3)}R$ a potential in the variables β_+, β_-, θ, φ, and ϑ. The reader should thus note the relevance of (2.33) for obtaining the above expressions. One might have expected it to introduce more complications than just using $\mathsf{p}_{ij} \equiv \mathrm{e}^{\beta}{}_{is}\pi^s{}_t\mathrm{e}^{-\beta}{}_{tj}$, but in fact we find just the opposite.

2.3.2 Momentum Constraint and Coordinate Conditions

However, one element seems to have been neglected in this analysis. In fact, we still have to solve the $\mathcal{H}^i = 0$ constraints, which would assist in eliminating three variables among β_+, β_-, θ, φ, and ϑ. For that purpose, there are two possibilities:

(i) we can solve $\mathcal{H}^i = 0$ and substitute the solutions into the action, or
(ii) we can deal with a constrained Hamiltonian description.

Associated with (i) is the issue of imposing a choice of coordinate conditions. So far, only one coordinate choice has been made, viz., $\mathcal{N} = 12\pi\zeta^3 H^{-1}_{\mathrm{ADM}}\mathrm{e}^{-3\Omega}$, and all variables in H_{ADM} are functions of Ω (see (2.29)). Since the spatial coordinates are not involved, their choice has remained quite arbitrary. This means that there still remains some freedom in adopting the $\mathcal{N}^i$ for the Bianchi model:

- We can choose the coordinate condition[20] $\mathcal{N}^i = 0$, or
- solve $\mathcal{H}^i = 0$ as indicated (a detailed discussion can be found in [43]).

---

[18] Having written $\mathsf{p}_{ij} = R^{-1}\mathsf{p}_\varepsilon\alpha_\varepsilon R$, $\varepsilon \equiv \{+, -, \theta, \varphi, \vartheta\}$, for suitable matrices α_ε.

[19] H_{ADM} is a 'square root' Hamiltonian. See Sect. 2.4 of this volume, and Exercise 7.1 of Vol. II.

[20] Leading to a complete Hamiltonian system with $\mathcal{H}^i = 0$ as ancillary conditions.

The simplest choice in cosmology is indeed to have $\mathcal{N}^i = 0$, although other options have been explored [43] in the context of, e.g., cosmologies with spatial rotation and specific matter content. But is $\mathcal{N}^i = 0$ consistent, our fellow explorer may ask, when we are investigating routes to quantum gravity?

Note 2.7 $\mathcal{N}^i = 0$ can lead to an *over* simplified choice. It means taking a foliation of $\mathcal{M}$ by spatial hypersurfaces Σ_t, and choosing *Gaussian normal coordinates*. The advantage is that the metric becomes

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j \,, \tag{2.39}$$

where $\mathcal{N}^i = 0$ indicates that we are using comoving coordinates and the proper time t ($\mathcal{N} = 1$), with $K_{ij} = -\dot{h}_{ij}$, where dots over symbols denote differentiation with respect to t. However, it is important to maintain consistency. Let us be more precise. In the ADM procedure, when implementing coordinate conditions, we can either choose $\mathcal{N}$ and $\mathcal{N}^i$ such that we fix the coordinate system or establish the coordinates and therefore determine $\mathcal{N}$ and $\mathcal{N}^i$ subsequently, requiring that in variational terms the system of equations will correspond to Einstein's equations. But it is possible, in spite of getting a canonical system, that these equations might not be equivalent to Einstein's equations. Depending on the case, some care may be needed, in particular to avoid carrying out the whole procedure, but leaving some substitutions, like those leading to (2.39), to later stages.

One argument in favour of this choice is that $\mathcal{H}^i = 0$ is *identically satisfied* for some Bianchi types, so that one can then insert, e.g., $\theta = \varphi = \vartheta = 0$, and likewise for their conjugate momenta, thereby setting $\mathcal{N}^i = 0$ with impunity. This is the case for diagonal Bianchi I, II, VIII and IX. (See [81, 82] for a thorough discussion on more Bianchi cases. The case of Bianchi class B seems less clear.)

The ADM formalism for diagonal Bianchi models then leads to

$$H^2_{\mathrm{ADM}} = \mathsf{p}_+^2 + \mathsf{p}_-^2 - 24\pi^2 h\,{}^{(3)}R \,, \tag{2.40}$$

corresponding to the Hamiltonian of the universe (as a point particle) moving in a *two*-dimensional (β_+, β_-) plane, with a *time-dependent* (ADM) potential

$$h\,{}^{(3)}R \equiv -\zeta^4 e^{-4\Omega}[V(\beta_+, \beta_-) - 1] \,, \tag{2.41}$$

with which the point universe may collide and bounce back. Here, $V(\beta_+, \beta_-)$ characterizes the Bianchi type, each with exponentially steep walls in β_+, β_-.

2.4 ADM (Hamiltonian) Quantization

It is worth stressing the fact that the ADM method involves solving the constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}^i = 0$ *classically*, to achieve a canonical action for the gravitational sector, *before* any quantization step is applied only to the 'true' degrees of freedom.

Note 2.8 The Dirac method retains $\mathcal{H}_\perp = 0$ and $\mathcal{H}^i = 0$, subsequently formulating them within a quantum operator framework. Since they differ in this way, the ADM and Dirac methods will determine different characteristics for the quantum states of the universe. The Bianchi IX model with rotation illustrates this.

But a generic spacetime manifold is not covered by a single coordinate chart. The resulting H_{ADM} may not then be self-adjoint, and it may not be possible to construct states for the system [79, 80].

It is also important to observe (see, e.g., (2.47) below) that the ADM reduction induces a Schrödinger equation

$$\mathrm{i}\frac{\partial \Psi(\Omega, \beta_\pm)}{\partial \Omega} = \mathcal{F}\left[-\mathrm{i}\frac{\partial}{\partial \beta_\pm}, \beta_\pm, \Omega\right]\Psi(\Omega, \beta_\pm) \ , \tag{2.42}$$

where $\mathcal{F}$ denotes a functional, in contrast to the procedure introduced by Dirac, which leads to the well known Wheeler–DeWitt equation (see Sect. 2.5). In some sense, the ADM formulation offers a glimpse of a *square root* of the Wheeler–DeWitt equation (see Chap. 4). Quantization of the action (2.36) and Hamiltonian (2.38) in the usual canonical approach requires

$$H_{\mathrm{ADM}} \to -\mathrm{i}\frac{\partial}{\partial \Omega} \ , \qquad \mathsf{p}_\pm \to -\mathrm{i}\frac{\partial}{\partial \beta_\pm} \ , \tag{2.43}$$

$$\mathsf{p}_\theta \to -\mathrm{i}\frac{\partial}{\partial \theta} \ , \qquad \mathsf{p}_\varphi \to -\mathrm{i}\frac{\partial}{\partial \varphi} \ , \qquad \mathsf{p}_\vartheta \to -\mathrm{i}\frac{\partial}{\partial \vartheta} \ . \tag{2.44}$$

With these differential operators, we find a wave function of the form

$$\Psi\big[\Omega, \beta_+, \beta_-, \theta, \varphi, \vartheta\big] \ , \tag{2.45}$$

which we explore in the following.

2.4.1 Bianchi I Models

For the *empty* diagonal Bianchi I case, we have $\omega^i = \mathrm{d}x^i$, $\mathcal{H}^i = 0$ is identically satisfied, ${}^{(3)}R = 0$, and we can put $\theta = \varphi = \vartheta = 0$ (likewise for their conjugate momenta), setting $\mathcal{N}^i = 0$ with impunity. The ADM Hamiltonian is

$$H_{\mathrm{ADM}}^2 = \mathsf{p}_+^2 + \mathsf{p}_-^2 \,, \tag{2.46}$$

and the corresponding classical analysis can be found in [43].

For the case of (2.46), a few approaches have been considered for the quantum ADM setting:

- The square root approach (Bethe, Schweber, de Hoffman).
- The Dirac matrix approach.
- The Schrödinger–Klein–Gordon approach.

The approach of Bethe, Schweber, and de Hoffman involves writing $H_{\mathrm{ADM}} = \left(\mathsf{p}_+^2 + \mathsf{p}_-^2\right)^{1/2}$, i.e., employing the differential equation

$$\mathrm{i}\frac{\partial \Psi}{\partial \Omega} = \pm \left(-\frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} \right)^{1/2} \Psi \,, \tag{2.47}$$

whereas in the Schrödinger–Klein–Gordon approach, we have to deal with H_{ADM}^2 instead, i.e.,

$$-\frac{\partial^2 \Psi}{\partial \Omega^2} + \frac{\partial^2 \Psi}{\partial \beta_+^2} - \frac{\partial^2 \Psi}{\partial \beta_-^2} = 0 \,. \tag{2.48}$$

The Dirac matrix approach is perhaps the most interesting within the context of this book. It will lead to a linear two-component spinor-type equation, which we know today how to interpret[21] but which remained quite uncharted up until the early 1970s. The essential point is that the solutions for the three approaches above are in the form of plane waves

$$\Psi = A\mathrm{e}^{\mathrm{i}(p_+\beta_+ + p_-\beta_- - E\Omega)} \,, \tag{2.49}$$

with A, p_+, p_-, $E = \pm\left(p_+^2 + p_-^2\right)^{1/2}$ as constants, labelling expanding and contracting universes without avoiding any singularity.

2.4.2 Friedmann–Robertson–Walker (FRW) Universe

The FRW cosmology in the ADM framework requires $\theta = \varphi = \vartheta = \beta_+ = \beta_- = 0$, which implies that *all* dynamical variables disappear, since

$$H_{\mathrm{ADM}}^2 = -24\pi^2 h \, {}^{(3)}\!R \,, \tag{2.50}$$

[21] The reader might briefly consult Sect. 7.1 of Vol. II on the matrix approach to $N = 1$ SUGRA cosmology.

with all space constraints identically satisfied (so we can take $\mathcal{N}^i = 0$ safely). Classically, the ADM Hamiltonian problem is solved with $H_{\rm ADM}$ as function of Ω, integrating

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{1}{\mathcal{N}} = \frac{H_{\rm ADM} e^{3\Omega}}{12\pi\zeta^3}\,, \tag{2.51}$$

to find Ω as a function of t. Matter in the form of a perfect fluid (adding *no* new degrees of freedom) requires

$$H^2_{\rm fluid} \sim \mathrm{e}^{-3(1-\gamma)\Omega}\,, \tag{2.52}$$

$$H^2_{\rm ADM} = H^{2(\rm empty)}_{\rm ADM} - 24\pi^2\sqrt{h}H^2_{\rm fluid}\,, \tag{2.53}$$

changing the constraints into $\mathcal{H}_\perp + \mathcal{H}_{\perp \rm m} = 0$ and $\mathcal{H}^i + \mathcal{H}^i_{\rm m} = 0$, where $\mathcal{H}_{\perp \rm m}$ and $\mathcal{H}^i_{\rm m}$ denote the matter contributions. This issue is of particular relevance, e.g., in *non*-diagonal Bianchi IX metrics.

In the Schrödinger–Klein–Gordon approach, the differential equation is

$$-\frac{\partial^2\Psi}{\partial\Omega^2} = \left[36\pi^2\zeta^4\mathrm{e}^{-4\Omega} + 384\pi^3\zeta^{3(\gamma-2)}\mathrm{e}^{-3(2-\gamma)\Omega}\right]\Psi\,, \tag{2.54}$$

and solutions (for the $k = 1$ *empty* case) are of the form

$$\Psi \simeq \lim_{m\to 0} K_{\mathrm{i}m/2}\left[\frac{1}{2}\mathrm{e}^{-2\Omega}\right]\,, \tag{2.55}$$

where $K_{\mathrm{i}m/2}$ is a modified Bessel function.

The Schrödinger equation for FRW with a positive cosmological constant, i.e., the de Sitter (dS) model, is

$$\mathrm{i}\frac{\partial\Psi}{\partial\Omega} = \pm\left(\lambda\mathrm{e}^{6\Omega} - k\mathrm{e}^{4\Omega}\right)^{1/2}\Psi\,, \tag{2.56}$$

where λ is the cosmological constant. However, $\left(\lambda\mathrm{e}^{6\Omega} - k\mathrm{e}^{4\Omega}\right)^{1/2}$ is a non-self-adjoint operator. Solutions are in the form of exponentials:

$$\Psi \simeq \exp\left[\mathrm{i}\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right)\right]\,, \quad a \sim \mathrm{e}^{\Omega} \to \infty\,, \quad z = (4\lambda)^{-1/3}(1-\lambda a^2)\,. \tag{2.57}$$

For a massless scalar field in a FRW universe, making the gauge choice $\Omega = t$, the result is

$$\mathrm{i}\frac{\partial\Psi}{\partial\Omega} = \pm\left(\hat{p}^2 + c_1\mathrm{e}^{c_2\Omega}\right)^{1/2}\Psi\,, \tag{2.58}$$

where $\hat{p}$ is a constant of integration. Using specific techniques such as spectral decomposition [79, 80], the term $\left(\hat{p}^2 + c_1 \mathrm{e}^{c_2 \Omega}\right)^{1/2}$ can be dealt with, and H_{ADM} is self-adjoint for $c_1 > 0$. Solutions are a superposition of modes, each of the form

$$f_p\left(\Omega, \phi\right) \simeq \exp\left[\mp \int^{\Omega} \left(\hat{p}^2 + c_1 \mathrm{e}^{c_2 \Omega}\right)^{1/2} \mathrm{d}x + \mathrm{i}\hat{p}\phi\right] , \qquad (2.59)$$

whose asymptotic behavior in some specific combinations is also found in solutions within the Dirac (Wheeler–DeWitt equation) approach, when $\Omega \to \infty$.

2.4.3 Bianchi IX Cosmologies

A vast bibliography is available for the Bianchi IX case, where rather important results have been extracted, ranging from the isotropization of Bianchi IX into a closed FRW model, up to the issue of chaos near the singularity (see, e.g., the *mixmaster* scenario[22] [60, 49, 62, 43]). In brief, for the diagonal case, the matrix β is associated with a diagonal metric, and taking $\theta = \varphi = \vartheta = 0$, all space constraints are identically satisfied (i.e., we can set $\mathcal{N}^i = 0$ with impunity). Then

$$H_{\mathrm{ADM}}^2 = \mathsf{p}_+^2 + \mathsf{p}_-^2 + 36\pi^2 \zeta^{-2} \mathrm{e}^{-4\Omega} \left[V(\beta_+, \beta_-) - 1\right] . \qquad (2.60)$$

Quantum mechanically, the Schrödinger–Klein–Gordon approach produces

$$-\frac{\partial^2 \Psi}{\partial \Omega^2} + \frac{\partial^2 \Psi}{\partial \beta_+^2} - \frac{\partial^2 \Psi}{\partial \beta_-^2} + \mathrm{e}^{-4\Omega}\left[V(\beta_+, \beta_-) - 1\right]\Psi = 0 , \qquad (2.61)$$

while for the Dirac matrix procedure, a linear set of equations in $-\mathrm{i}\partial/\partial\beta_\pm$ is obtained, but also involving derivatives of $V(\beta_+, \beta_-)$ which apparently cannot be removed algebraically (see Chap. 7 of Vol. II).

In the *symmetric* case, we have $\theta = \vartheta = 0$, $\mathsf{p}_\theta = \mathsf{p}_\vartheta = 0$, but $\varphi \neq 0$ and we *cannot* take $\mathcal{H}^i$ to be identically zero, because this would force $\mathsf{p}_\varphi = 0$, implying φ constant (i.e., $\varphi = 0$). This means that some matter must be added to satisfy the constraints, e.g., in the form $\mathcal{H}^i + \mathcal{H}^i_{\mathrm{fluid}} = 0$. Moreover, we find that $\mathsf{p}_\phi = 32\pi^2 \mu C$, where μ and C are constants, and the Hamiltonian is cyclic in φ. The ADM Hamiltonian is

[22] This is a suitable prototype for analysing the initial singularity. As two spatial directions approach spatial collapse, this is prevented by the other direction being stretched, within a dynamical setting where the curvature is the determining factor. The process repeats itself infinitely many times in a quasi-periodic, but *not* periodic manner.

$$H_{\mathrm{ADM}}^2 = \mathsf{p}_+^2 + \mathsf{p}_-^2 + \frac{3\mathsf{p}_\varphi^2}{\sinh^2 2\sqrt{3}\beta_-} + 36\pi^2 a_0^4 \mathrm{e}^{-4\Omega}\left[V(\beta_+,\beta_-) - 1\right]$$
$$+384\pi^3 a_0^3 \mathrm{e}^{-3\Omega}\mu\left(1 + C^2 a_0^{-2}\mathrm{e}^{2\Omega+4\beta_+}\right)^{1/2} . \quad (2.62)$$

The constraint $\mathsf{p}_\varphi = 32\pi^2\mu C$ can be inserted, but we may leave the ADM Hamiltonian as it is and deal with it later. In the latter, the differential equation is

$$-\frac{\partial^2\Psi}{\partial\Omega^2} + \frac{\partial^2\Psi}{\partial\beta_+^2} - \frac{\partial^2\Psi}{\partial\beta_-^2} + \left\{\frac{3\,(\mu C)^2}{\sinh^2 2\sqrt{3}\beta_-} + \mathrm{e}^{-4\Omega}\left[V(\beta_+,\beta_-) - 1\right]\right. \quad (2.63)$$
$$\left. + \,\mathrm{e}^{-3\Omega}\mu\left[1 + (2C)\,\mathrm{e}^{2\Omega+4\beta_+}\right]^{1/2}\right\}\Psi = 0\,,$$

and in the former, when we quantize *before* imposing all the constraints,

$$-\frac{\partial^2\Psi}{\partial\Omega^2} + \frac{\partial^2\Psi}{\partial\beta_+^2} - \frac{\partial^2\Psi}{\partial\beta_-^2} + 3(\sinh 2\sqrt{3}\beta_-)^{-2}\frac{\partial^2\Psi}{\partial\varphi^2} \quad (2.64)$$
$$+\left\{\mathrm{e}^{-4\Omega}\left[V(\beta_+,\beta_-) - 1\right] + c_1\mathrm{e}^{-3\Omega}\left(1 + 4c_2^2\mathrm{e}^{2\Omega+4\beta_+}\right)^{1/2}\right\}\Psi = 0\,,$$

where c_1 and c_2 are constants. We direct the reader to Exercise 2.2 for additional remarks.

2.5 Superspace and the Dirac Quantization Method

In Dirac's approach to quantum gravity, we do not solve all constraints prior to quantization to identify the physical degrees of freedom. The Einstein equations are obtained by varying $\mathcal{N}\mathcal{H}_\perp + \mathcal{N}_i\mathcal{H}^i = 0$. The constraints are then used to eliminate all the non-dynamical degrees of freedom from the solutions to the variational equations, yielding[23] only two, viz., h_{ij} and π^{ij}, at each space point.

In this context, a relevant configuration space has been introduced. This is (the gravitational) *superspace* **S** [56–59]:

- Its main element is the space of all Riemannian 3-metrics and matter configurations on the spatial hypersurfaces Σ_t:

$$\mathrm{Riem}(\Sigma) \equiv \left\{h_{ij}(x), \phi(x)\,|\,x \in \Sigma_t\right\} . \quad (2.65)$$

- Riem (Σ) constitutes an infinite-dimensional space, with a finite number of degrees of freedom at each point $x \in \Sigma_t$.

[23] There has nevertheless been some debate. See Sect. 4.2.3 of [41].

- Configurations which can be related to each other by a spatial diffeomorphism $\mathrm{Diff}_0(\Sigma)$ are equivalent (common intrinsic geometry) and considered to be the *same*.
- Hence, the physical configuration space **S** designated as *superspace* is given by

$$\frac{\mathrm{Riem}\,(\Sigma)}{\mathrm{Diff}_0(\Sigma)}\,, \tag{2.66}$$

where the subscript zero denotes spatial diffeomorphisms which are connected to the identity.
- In brief, superspace is defined as the space of all Riemannian metrics Riem (Σ) modulo all possible transformations which give the same geometry by means of a different but related metric.[24]

Superspace admits a metric, allowing rather important properties to be retrieved, within, e.g., the context of Einstein's general relativity. It is obtained (see [41] for more details on this feature) from the DeWitt metric (2.25), which is the metric for Riem (Σ). The DeWitt metric can be written as

$$\mathcal{G}_{XY}(x) \equiv \mathcal{G}_{(ij)(kl)}(x)\,, \tag{2.67}$$

with

$$X, Y \in \{h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}\}\,. \tag{2.68}$$

It has signature $(-+++++)$ at each point. The *full* superspace incorporates the matter degrees of freedom. One way of adding these is to define $\mathcal{G}_{\phi\phi}(x)$.

In the superspace context, general relativity can be interpreted as providing the evolution of a 3-geometry in a timelike direction, whose meaning is of course dependent on how one implements the $3+1$ decomposition of 4D spacetime. In other words, general relativity enables a description of how 3-metrics evolve, considering each evolution of the metric as a *trajectory* in superspace (which can even be interpreted in a geodesic context), and determined by the choice of timelike coordinate. In fact, the constraint $\mathcal{H}_\perp = 0$ is equivalent to an equation of motion for h^{ij} in minisuperspace [75].

Canonical quantization goes as follows[25]:

1. We introduce the wave function of the universe $\Psi[h_{ij}, \phi]$ as a functional[26] on Riem (Σ), namely, the amplitude for a given 3-geometry with ϕ defined on it. A more detailed description will be provided in Note 2.13.

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[24] The subspace of superspace associated with all homogeneous 3-cosmologies, which is therefore finite-dimensional, is called *mini*superspace. It lies at the core of quantum cosmology (see Sect. 2.7).

[25] For simplicity, the matter sector is reduced to a single scalar field.

[26] Note that Ψ does not depend explicitly on time, as general relativity is time-reparametrisation invariant. Time is present *implicitly* in the dynamical variables h_{ij} and ϕ. This is the origin of the *time problem* in quantum gravity (see [41] and references therein).

2. The canonical momenta become operators via

$$\pi^{ij} \longrightarrow -\mathrm{i}\frac{\delta}{\delta h_{ij}}\,, \quad \pi_\phi \longrightarrow -\mathrm{i}\frac{\delta}{\delta\phi}\,, \quad \pi^0 \longrightarrow -\mathrm{i}\frac{\delta}{\delta\mathcal{N}}\,, \quad \pi_i \longrightarrow -\mathrm{i}\frac{\delta}{\delta\mathcal{N}_i}\,. \tag{2.69}$$

3. The wave function is annihilated by the operator constraints (see Appendix B):
   a. The *primary* constraints mean that Ψ is independent of $\mathcal{N}$ and $\mathcal{N}^i$:

$$\pi^0\Psi = -\mathrm{i}\frac{\delta\Psi}{\delta\mathcal{N}} = 0\,, \qquad \pi^i\Psi = -\mathrm{i}\frac{\delta\Psi}{\delta\mathcal{N}_i} = 0\,. \tag{2.70}$$

   b. The *secondary* momentum constraint becomes (with orthonormal frame components)

$$\mathcal{H}^i\Psi = 0 \Longrightarrow -2\mathrm{i}\left[\frac{\delta\Psi}{\delta h_{ij}}\right]_{|j} = \sqrt{h}\mathsf{k}^2 T^{\perp i}\Psi\,, \tag{2.71}$$

   ensuring that Ψ is invariant under coordinate $\{h_{ij}(x), \phi(x)\}$ transformations in the spatial hypersurface (see Exercise 2.3).
   c. The Hamiltonian constraint is

$$\mathcal{H}_\perp\Psi = \left\{-2\mathsf{k}^2\mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{\sqrt{h}}{2\mathsf{k}^2}\left[-{}^{(3)}R + 2\Lambda + 2\mathsf{k}^2T^{\perp\perp}\right]\right\}\Psi = 0\,, \tag{2.72}$$

   with, e.g.,

$$T^{\perp\perp} = -\frac{1}{2h}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}h^{ij}\phi_{,i}\,\phi_{,j} + V(\phi)\,, \tag{2.73}$$

   known as the *Wheeler–DeWitt equation* [49, 58].

Note 2.9 Strictly speaking, the constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}^i = 0$ should read $\mathcal{H}_\perp \cong 0$ and $\mathcal{H}^i \cong 0$, i.e., meaning that they vanish as constraints, being set to zero *only after* all Poisson brackets have been computed. However, for the (computational) context of SQC, treating them as equalities is quite adequate (see Appendix B and Note 2.4).

Note 2.10 The Wheeler–DeWitt equation constitutes a second-order functional differential equation on superspace, i.e., *one* differential equation at *each* point $x \in \Sigma_t$. Factor-ordering ambiguities are present and several suggestions exist on how to deal with this (note that the ADM quantization has no ambiguities of this type [43]), e.g., for which the derivatives become a Laplacian in the DeWitt supermetric.

Note 2.11 $\mathcal{H}^i\Psi[^{(3)}h] = 0$ represents invariance under spatial diffeomorphisms (see Exercise 2.3), while $\mathcal{H}_\perp\Psi[^3h] = 0$ is the Hamiltonian constraint (yielding the Wheeler–DeWitt equation). The latter is more difficult to deal with, as it relates to the generation of time coordinate transformations and also the dynamical evolution from the initial hypersurface Σ_t. It is a hyperbolic operator, and this precludes straightforward use of the usual probabilistic interpretation of quantum mechanics [41].

2.6 Algebra of Constraints

As indicated in Appendix B, the presence of secondary constraints in a theory (with only first class constraints) means that they also constitute a set of generators for transformations that leave the action and equations of motion of the theory *invariant*. In other words, the presence of such constraints displays the symmetry properties of the system, when employing a Hamiltonian description.[27]

Hence, the set (2.70), (2.71), and (2.72) constitutes more than just quantum equations for the states of the system to satisfy. They convey the intrinsic physical meaning and properties of the system under investigation, and should be properly discussed. Concerning general relativity, it is important to say a few words in this section, as it will become of relevance when moving towards SUGRA, where general relativity is a specific limit, and therefore wider invariance properties will be present,[28] which deserve to be adequately explored (see Chap. 4). Although restricted to pure general relativity, the rest of this section constitutes a preparation for elements to be introduced and explored in Chap. 4.

The essential feature is that a Hamiltonian description of a theory, which gives the evolution of initial data from one arbitrary spacelike hypersurface to another, involves arbitrary 3D hypersurfaces embedded in a 4D space. Let us therefore take two *generically* imbedded (and labelled) 3D surfaces (see Fig. 2.2) $\Sigma^\mu(x^i)$ and $\Sigma'^\mu(x^i) = \Sigma^\mu(x^i) + \delta\xi^\mu(x^i)$ [83]. In a 4D Riemannian space, with signature $(-,+,+,+)$, we consider a normal $\mathbf{n} \rightarrow n^\mu$ to the surface, and three vectors $\mathbf{e}_i \rightarrow e^\mu{}_i$, $i = 1, 2, 3$, tangent to the surface (in essence, it is a triad of spatial basis vectors), together forming a *tetrad* basis such that $n^\mu n_\mu = -1$, $n_\mu e^\mu{}_i = 0$, and $e^\mu{}_i e_{\mu j} = h_{ij}$. We then have[29]:

---

[27] This statement, of course, assumes that we have a gauge theory, whose secondary constraints are also of first class, and hence represent the generators of gauge transformations.

[28] In particular, in the case where torsion is present through fermionic sectors (in particular, gravitinos (see Chap. 4)) and where the use of a tetrad representation allows one to bring in the Lorentz constraints as well (see Exercise 2.1).

[29] No strict assumptions are made about the metric structure of the 3D and 4D spaces (see also Fig. 2.1).

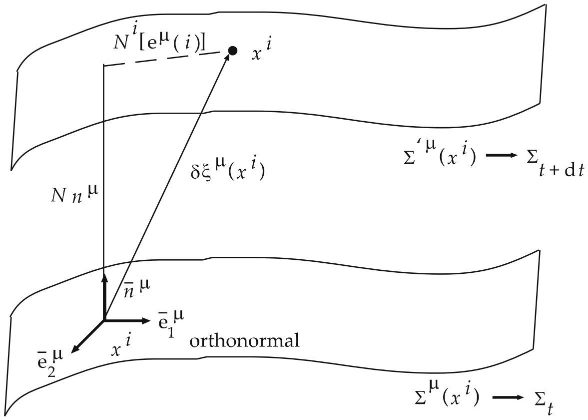

Fig. 2.2 A deformation ($\delta\xi^\mu$) connecting surface Σ^μ to surface Σ'^μ, and its (re)labeling with time t as parameter. The deformation becomes $\delta\xi^\mu = \mathcal{N}^\mu \delta t$, while $\overline{n}^\mu$ has become $(1, 0, 0, 0)$ in components, but corresponds in Fig. 2.1 to $(\mathcal{N}, 0, 0, 0)$ and the vector $\delta\xi^\mu = \mathcal{N}\mathrm{d}t$

- The generators $\mathcal{H}^\mu$ of generic deformations give the change in a functional $\mathcal{F}(Q, P)$ of the *generic* canonical variables (Q, P) as a result of the deformation.
- This deformation is given by $\delta\mathcal{F}(x) = \int_\Sigma \mathrm{d}^3x' \big[\mathcal{F}(x), \mathcal{H}_\mu(x')\big] \delta\xi^\mu(x'^i)$.
- If the spacelike surfaces are instead labelled by t, the deformation (see Fig. 2.2) becomes $\delta\xi^\mu(x^i) = \mathcal{N}^\mu \delta t$, with the Hamiltonian given by (see (2.22))

$$H = \int \mathrm{d}^3x \mathcal{N}^\mu \mathcal{H}_\mu = \int \mathrm{d}^3x (\mathcal{N}\mathcal{H}^\perp + \mathcal{N}^i \mathcal{H}_i) ,$$

  The arbitrary $\mathcal{N}^\mu \equiv \left\{\mathcal{N}, \mathcal{N}^i\right\}$ is then provided by the 4-metric representation in (2.8).
- It is in this basis that the generators of deformations become $\mathcal{H}_\perp = \mathcal{H}_\mu n^\mu$ and $\mathcal{H}_i = \mathcal{H}_\mu e^\mu{}_i$, representing orthogonal and tangential deformations, respectively.

Therefore, for the above Riemannian geometry and assuming path independence, i.e., assuming that the way the functional $\mathcal{F}$ varies from $\Sigma^\mu(x^i)$ to $\Sigma'^\mu(x^i)$ is *not* related to the process between them, this implies the following algebra [84, 85] for the generators:

$$\Big[\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\Big] = \Big[h^{ij}(x)\mathcal{H}_j(x) + h^{ij}(x')\mathcal{H}_j(x')\Big]\delta_i(x, x') , \qquad (2.74)$$

$$\Big[\mathcal{H}_\perp(x), \mathcal{H}_j(x')\Big] = \mathcal{H}_\perp(x')\delta_j(x, x') , \qquad (2.75)$$

$$\Big[\mathcal{H}_i(x), \mathcal{H}_j(x')\Big] = \mathcal{H}_i(x')\delta_j(x, x') - \mathcal{H}_j(x')\delta_i(x, x') , \qquad (2.76)$$

with $\mathcal{H}_\mu = 0$ (see also Note 2.9).

Note 2.12 The 3-metric h_{ij} becomes of relevance, as (2.74) indicates. In fact, it has been shown [84, 85] that the theory with (2.74), (2.75), and (2.76) and canonical variables $\left(h_{ij,}\pi^{ij}\right)$ satisfying

$$\left[h_{ij}(x), \pi^{kl}(x')\right] = \frac{1}{2}\left(\delta^k{}_i\delta^l{}_j + \delta^k{}_j\delta^l{}_i\right), \tag{2.77}$$

is indeed general relativity. In other words, when we look for the corresponding $\mathcal{H}_\perp$ and $\mathcal{H}_i$, the result is (2.23) and (2.24), respectively. General relativity is thus retrieved as the theory for the metric field in a Riemannian spacetime!

So why the emphasis on the above note, in the context of this book? The eager explorer would be quite right to raise the question. The point to emphasize is the connection, which only the Hamiltonian perspective provides, between (a) the geometry of the spacetime and (b) the algebra of the deformation generators, which thus convey the geometry. This encourages us to investigate what would change if the assumption of a Riemannian geometry were modified e.g., by taking expressions that differ from (2.74), (2.75), and (2.76), and in particular, by taking a SUGRA background (see Chap. 4).

But instead of proceeding directly along that road, there is time for a brief comment. In SUGRA, there are *extra* terms in the deformation generators that imply a much more complicated geometry than the Riemannian geometry of general relativity. Moreover, the relation between (a) the spinor structure e.g., in terms of Dirac matrices and bilinear invariants (see Chap. 3), and (b) the geometric properties, i.e., the curvature tensor, is not yet fully understood. In other words, an intriguing route to explore is to proceed from a set of such (deformation) generators and their algebra to retrieve the not yet totally charted geometry associated with SUGRA, where both the metric *and* the gravitino (Rarita–Schwinger) fields are of equivalent influence (see also Sect. 3.4.3 of Vol. I and Sect. 4.2 of Vol. II).

2.7 Boundary Conditions

The framework introduced in Sects. 2.5 and 2.6, and also Sect. 2.4, is indeed quite elegant and promising. Some results are particularly relevant. But, as our diligent reader may be thinking, in the end, the essential question is the following: How can we make any prediction with the above structure, in order to study the physical consequences for the evolution of the universe from an initial quantum stage? These are as yet *open* questions, in the sense that research is active, and some elements can be found in Chap. 2 of Vol. II. For the moment, we will describe the background ingredients, namely, how to get solutions with physical meaning in order to proceed to make predictions.

The nature of the Wheeler–DeWitt equation requires the implementation of specific boundary conditions for the wave function of the universe. The Hamiltonian formulation of canonical general relativity, following ADM plus Dirac quantization guidelines, were the focus of attention in the 1960s–1970s, while the consequences of boundary (and initial!) conditions became the line of sight for navigating into quantum gravity from the 1980s onwards.

Two proposals for boundary conditions[30] have attracted most attention: the *no-boundary condition* due to J. Hartle and S. Hawking [86, 27, 28] and the *tunneling condition* due to A. Vilenkin [31, 32, 35–37]. In fact, and with the exception of specific cases of simple situations such as particular minisuperspaces (see the next section), a choice of boundary condition is mandatory to solve the Wheeler–DeWitt equation. However, as the attentive explorer is about to point out, we also seem to require an additional element in the form of a *new fundamental law of physics*, which will *select* the boundary condition. But how should we do this [21–24]? In other known physical situations, the boundary conditions are obvious and follow, for example, from the symmetries of the system. Moreover, it seems that implementing a boundary condition for the Wheeler–DeWitt equation is just a slightly obscure way of distracting attention from (but not really solving) the issue of the arbitrary initial choice of *other* parameters, which provide the choice of the classical evolution, by leading the discussion into the choice of parameters to describe its quantum evolution. But this is not so. *If* quantum mechanics is the fundamental framework for physical interactions, then quantum dynamics precedes the classical dynamics and it must be dealt with first.

2.7.1 The No-Boundary Proposal

Let us start with the proposal made by J. Hartle and S. Hawking, which includes the (Feynman) path integral method as a pertinent tool [25, 26, 61].

Note 2.13 In brief, for the path integral:

- The amplitude to proceed from one state, e.g., $|h_{ij}, \varphi; \Sigma_t\rangle$, describing the system with an intrinsic 3-metric h_{ij} and matter configuration φ on an initial hypersurface Σ_t, to another state $|h'_{ij}, \varphi'; \Sigma'_{t'}\rangle$, is given by a *functional* integral of $\mathrm{e}^{\mathrm{i}S}$ over all 4-geometries $g_{\mu\nu}$ and matter configurations ϕ which interpolate between the initial and final configurations:

$$\left\langle h'_{ij}, \varphi', \Sigma'_{t'} | h_{ij}, \varphi, \Sigma_t \right\rangle = \sum_{\mathcal{M}} \int \mathcal{D}g\, \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S[g_{\mu\nu},\phi]} . \tag{2.78}$$

[30] Other proposals have been put forward: the infinite-wall condition due to B. DeWitt [43], the all-possible-boundaries condition due to Suen and Young [87], and the symmetric initial condition due to Conradi and Zeh [88].

- One possible procedure is to replace S by the corresponding Euclidean action $I[g_{\mu\nu}, \phi] \equiv -\mathrm{i}S[g_{\mu\nu}, \phi]$. The sum is taken over all 4-metrics with signature $(+ + ++)$, which induce the appropriate 3-metrics h_{ij} and h'_{ij} on hypersurfaces. In essence, this is a *Wick rotation* to imaginary time $t \to -\mathrm{i}\tau$.
- The wave function Ψ of the universe on a hypersurface Σ_t, with intrinsic 3-metric h_{ij} and matter configuration φ, is then

$$\Psi\left[h_{ij}, \varphi; \Sigma'_{t'}\right] = \sum_{\mathcal{M}} \int \mathcal{D}g\, \mathcal{D}\phi\, \mathrm{e}^{-I[g_{\mu\nu},\phi]} . \tag{2.79}$$

  - This approach has had some significant successes in the context of quantum gravity.
  - The motivation is that, in flat spacetime, the integral is oscillatory, therefore not convergent. The solution, which extremises the action, requires one to solve a hyperbolic equation, having either no solutions or an infinite number.
  - In terms of $I = -\mathrm{i}S$, the action becomes positive-definite, the path integral is exponentially damped, and convergence *may* be possible (non-gravitationally speaking), with what is now an elliptic equation and well-posed boundary conditions.
  - But some fundamental and serious problems persist: (a) The Euclideanized gravitational action is not positive-definite, so the path integral does *not* converge if the sum is restricted to real Euclidean-signature metrics. (b) *Complex* metrics must be included. (c) There is *no unique* contour to integrate along in superspace. (d) The result *may* depend crucially on the contour that is chosen [20, 89–91, 38]. (e) The measure in it must be properly defined, something that has not yet been achieved.

- Another approach, still possibly requiring complex metrics in the sum, is to maintain the Lorentzian path integral, i.e., with $\mathrm{e}^{\mathrm{i}S}$ instead of e^{-I} in a definition for Ψ [31, 34]. A particular variant is [65] to Wick rotate with the opposite sign, i.e., $t \to +\mathrm{i}\tau$ instead of $t \to -\mathrm{i}\tau$, leading to a factor e^{+I} instead of e^{-I}. (A convergent path integral is obtained for the scale factor, but *not* for the matter or inhomogeneous modes of the spatial metric.)
- The pertinent feature is that wave functions defined according to the path integral procedure satisfy the Wheeler–DeWitt equation [92], provided that the action, the measure, and the class of paths summed over are invariant under diffeomorphisms.

The proposal due to Hartle and Hawking [86] is then to restrict the $\mathcal{M}$ over which the sum is taken in the definition of the wave function of the universe (2.79) to include *only* compact Riemannian 4-manifolds, where the spatial hypersurface Σ_t

on which Ψ is defined constitutes the *only* boundary, considering only matter configurations that are regular on these geometries. The universe then has *no* singular boundary to the past.[31]

The no-boundary proposal then informs as to what *initial* conditions should be laid down, at an initial time $\tau = 0$, on $h_{ij}(x, 0)$, $\varphi(x, 0)$, and their derivatives. Within a semiclassical approximation, we use $\Psi \simeq \mathrm{e}^{-I_{\mathrm{cl}}}$, where I_{cl} is the classical (possibly complex) action evaluated along the solution to the Euclidean field equations. In other words, the boundary conditions are at the *classical* level:

- The four-geometry is closed.
- The saddle points of the functional integral correspond to regular solutions of the classical field equations, consistent with the data on Σ.

2.7.2 The Tunneling Proposal

This is the alternative advanced by A. Vilenkin [31, 32, 35–37]. According to this, the boundary condition for the wave function Ψ should be that the universe *tunnels into existence from nothing*. No restrictions are made on the initial geometry (compare with the Hartle–Hawking proposal in Sect. 2.7.1).

One version of this defines the wave function by a functional path integral over Lorentzian metrics which interpolate between a given matter configuration φ and 3-geometry h_{ij}, and a *vanishing* 3-geometry to its past:

$$\Psi\left[h_{ij}, \varphi; \Sigma\right] = \sum_{\mathcal{M}} \int_0^{(h,\varphi)} \mathcal{D}g\, \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S[g_{\mu\nu},\phi]}\,, \tag{2.80}$$

which on superspace is as follows [20, 38]:

- Superspace has a boundary, consisting of 3-metrics and matter configurations for which the 3-curvature goes to infinity.
- It is essential to distinguish points on the boundary of superspace which correspond to real singularities of the 4-geometry from those that correspond to degenerate slicings.[32] The former is the *singular boundary* of superspace, and the latter the *non-singular boundary*.
- The tunneling proposal is that Ψ should be everywhere bounded and furthermore:
  - At singular boundaries of superspace, Ψ should include only outgoing modes, i.e., those that carry a flux *out* of superspace.
  - Ingoing modes should only enter at the non-singular boundary.

---

[31] In this manner, the initial singularity is somehow smoothed out, by using imaginary time. A surface with $\sqrt{h} = 0$ would be singular in a Lorentzian signature metric, but this may be quite different in the corresponding Riemannian geometry, as can be understood from the example of the 4-sphere S^4 [38].

[32] See the example of S^4 [20, 38]. Not all singular 3-geometries will correspond to singular 4-geometries. It is possible to obtain a singular 3-geometry by a degenerate slicing of the 4-geometry.

The attentive reader will already be asking whether there is in superspace a clear structure of Killing vectors, and hence of positive and negative (i.e., ingoing and outgoing) frequency modes. Or again whether there is an unambiguous way to split its boundary into singular and non-singular sectors. In short, how can this proposal be made more *computational*? And for this, minisuperspace provides an adequate exploring ground.

2.8 Minisuperspace

The elegance and richness of the superspace concept embodies a fundamental difficulty, namely, that superspace has infinitely many dimensions. In order to retrieve physical and predictive information, even if of a restricted range of validity, some researchers in the 1960s [49, 62] suggested truncating those degrees of freedom to a finite number. And so minisuperspace was born. The idea is to take only homogeneous metrics, which, for each point $x \in \mathbf{S}$, imply instead a finite number of degrees of freedom for a subset of superspace.

However, one needs to ask whether minisuperspace possesses any predictive meaning and usefulness concerning the quantization of gravity. In other words, can this truncation be justified? In minisuperspace, we simultaneously set most of the field modes and their conjugate momenta to zero, which violates the Heisenberg uncertainty principle. The validity of this approximation is indeed questioned in the literature [93–95]. However, the hope is that the minisuperspace approach will eventually be justified in some way, thereby providing a guiding light for probing through uncharted research territory. Minisuperspace models, displaying homogeneity, may constitute a reasonable framework from which to extend beyond current frontiers, broadening our understanding towards more complex descriptions, and eventually including some superspace-like features. This is *the* challenge, and to meet and deal with it, it is reasonable to probe and validate our methods in simpler testing grounds.

In minisuperspace, instead of a Wheeler–DeWitt equation for each point of the spatial hypersurface Σ_t, a *single* Wheeler–DeWitt equation is enough. For a FRW model, with a single scalar field, the corresponding minisuperspace is 2D with coordinates $q_X \equiv \{a, \phi\}$, $X = 1, 2$. For Bianchi models, whose 3-metric is of the form $h_{ij}\mathrm{d}x^i\mathrm{d}x^j \equiv h_{ij}(t)\omega^i \otimes \omega^j$, the minisuperspace can be anywhere between 2D and 5D (see Sect. 2.3). The intrepid explorer is invited to establish a relation between the Einstein equations and geodesics (Exercise 2.4), and discuss how to retrieve constants of motion in a minisuperspace (Exercise 2.5), in terms of the Misner–Ryan parametrization (Exercise 2.6), obtaining minisuperspace metrics with signature $(-+++++)$ (observing that some have zero, positive, or negative minisuperspace curvature in terms of the minisuperspace coordinates).

Quantization within minisuperspace, i.e., quantum cosmology essentially leads to the quantum mechanics of a specific constrained system, i.e., with time reparametrization invariance. From the action [using the notation of (2.68)]

$$S = \int \mathrm{d}t \left(\pi_X \dot{q}^X - \mathcal{N}\mathcal{H}_\perp \right) , \tag{2.81}$$

and Hamiltonian constraint

$$\frac{1}{2}\mathcal{G}^{XY}\pi_X\pi_Y + U(q) = 0 , \tag{2.82}$$

with the canonical momenta and Hamiltonian, respectively, given by[33]

$$\pi_X = \frac{\partial L}{\partial \dot{q}^X} = \frac{\mathcal{G}_{XY}\dot{q}^Y}{\mathcal{N}} , \tag{2.83}$$

$$H = \pi_X \dot{q}^X - L = \mathcal{N}\left[\frac{1}{2}\mathcal{G}^{XY}\pi_X\pi_X + U(q)\right] \equiv \mathcal{N}\mathcal{H} , \tag{2.84}$$

canonical quantisation yields

$$\mathcal{H}\Psi = \left[-\frac{1}{2}\nabla^2 + U(q)\right]\Psi = 0 . \tag{2.85}$$

This is the corresponding Wheeler–DeWitt equation, and

$$\nabla^2 \equiv \frac{1}{\sqrt{-\mathcal{G}}}\partial_X\left(\sqrt{-\mathcal{G}}\mathcal{G}^{XY}\partial_Y\right) \tag{2.86}$$

is the corresponding covariant Laplace–Beltrami operator of the *minisupermetric*. Note that the quantum mechanical formulation of the Wheeler–DeWitt equation [20, 41, 38] involves a choice of *factor ordering*, and the one in (2.85) is just one arbitrary and rather simplistic choice.

Note 2.14 Another particular choice requires a *conformally invariant* (in the sense of minisuperspace coordinate transformations) Wheeler–DeWitt equation, with

$$\mathcal{H}\Psi \equiv \left[-\frac{1}{2}\nabla^2 + \frac{\mathbf{D}-2}{8(\mathbf{D}-1)}\mathbf{R} + U(q)\right]\Psi = 0 , \tag{2.87}$$

where **D** is the dimension of the minisuperspace and **R** is the scalar curvature obtained from the minisupermetric.

[33] In minisuperspaces where $\mathcal{N}^i = 0$ is imposed, with $\mathcal{H}_i = 0$ identically satisfied (see Sects. 2.4.1 and 2.4.2), we can drop the $\perp$ in $\mathcal{H} = \mathcal{H}_\perp$.

2.8.1 The WKB Approximation

Solving the Wheeler–DeWitt equation for a minisuperspace model may require appropriate approximate solutions within the semiclassical limit [41]. Basically, we take the wave function to be of the form

$$\Psi \simeq \sum_n \Psi_n \equiv \sum_n \mathcal{A}_n e^{-I_n} , \tag{2.88}$$

where the sum is over the saddle points (found from the path integral description and neither purely real nor purely imaginary), and the $\mathcal{A}_n$ are prefactors. There will be regions in which the wave function is exponential, viz., $\Psi \simeq e^{-I}$, where I is the Euclidean action, and regions in which it is oscillatory, viz., $\Psi \simeq e^{iS}$, where S is the classical action. The reader is invited to confirm this in Exercise 2.7. The latter regions can be associated with classical (oscillatory) stages, while the former would correspond to classically inaccessible regions, e.g., tunneling regions.

The first-order WKB wave function

$$\Psi_n = C_n \exp\left(iS_n - \frac{1}{2} \int ds\, \nabla^2 S_n \right) \tag{2.89}$$

for minisuperspace quantum cosmology (see Exercise 2.7) is the analogue of a coherent state in quantum mechanics, peaked about a classical particle trajectory, and thus predicting (with some provisos) classical behaviour.

Note 2.15 In fact, for a superposition $\Psi = \sum_n \Psi_n$ of such states, interference between them may destroy the classical behaviour. A fully classical trajectory requires the quantum mechanical interference to become negligible, and this in turn requires a suitable *decoherence* mechanism. Subsequently, Ψ would predict a classical spacetime *if* a WKB-type solution induces a suitable correlation between π_X and $q^X = (a, \phi, \ldots)$ as in (C.48), through the use of quantum distribution functions, such as the Wigner function.

In essence, wave functions of the oscillatory type $\Psi \sim e^{iS}$ lead to a correlation between coordinates and momenta, whereas wave functions of the exponential type $\Psi \sim e^{I}$ do not. The latter correspond to stages of a purely quantum regime. Although we could explore this further, there is plenty of information regarding the issue of quantum decoherence in quantum cosmology [20, 41]. Decoherence is a necessary element of the WKB framework of quantum cosmology. Without it, chaotic cosmological solutions would lead to a breakdown of the WKB approximation.

2.8.2 Minisuperspaces and Boundary Conditions

A simple and illustrative case is the FRW universe with a single scalar field and a potential. As in [20, 38], we adopt $\varsigma^2 \equiv \mathsf{k}^2/{}^{(3)}\mathsf{V}$, where ${}^{(3)}\mathsf{V}$ is the 3-volume of the unit hypersurface,[34] and a $k = +1$ geometry. The metric (2.1) becomes

$$\mathrm{d}s^2 = \varsigma^2 \left\{ -\mathcal{N}^2 \mathrm{d}t^2 + a^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \right] \right\} . \tag{2.90}$$

In addition, the scalar field action becomes

$$S_\phi \simeq \frac{3}{\mathsf{k}^2} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{V(\phi)}{2\varsigma^2} \right] . \tag{2.91}$$

Basically, the following changes are introduced

$$\mathcal{N} \to \mathcal{N} , \quad a \to \varsigma a , \quad \phi \to \sqrt{3}\phi/\mathsf{k} , \quad {}^{(3)}V \to \frac{{}^{(3)}V}{2\mathsf{k}^2\varsigma^2} , \tag{2.92}$$

making a, ϕ, and ${}^{(3)}V$ now dimensionless.

Choosing the gauge $\mathcal{N}_i = 0$ and obtaining ${}^{(3)}R = 6k/\varsigma^2 a^2$, the action is of the form (2.21), with the minisupermetric given by ($\alpha \equiv \ln a$),

$$\mathcal{G}_{XY} \mathrm{d}q^X \mathrm{d}q^Y = -a \mathrm{d}a^2 + a^3 \mathrm{d}\phi^2 = \mathrm{e}^{3\alpha} \left(-\mathrm{d}\alpha^2 + \mathrm{d}\phi^2 \right) , \tag{2.93}$$

and the potential is

$$U = \frac{1}{2} \left[a^3 V(\phi) - ka \right] . \tag{2.94}$$

The canonical momenta are given by

$$\pi^0 = 0 , \quad \pi_a = -\frac{a\dot{a}}{\mathcal{N}} , \quad \pi_\phi = \frac{a^3 \dot{\phi}}{\mathcal{N}} , \tag{2.95}$$

and the classical Hamiltonian constraint is

$$\mathcal{H} = \frac{1}{2} \left(-\frac{a\dot{a}^2}{\mathcal{N}^2} + \frac{a^3 \dot{\phi}^2}{\mathcal{N}^2} - ka + a^3 V \right) = 0 . \tag{2.96}$$

The Wheeler–DeWitt equation is required to quantize this:

---

[34] Note that, for the 3-sphere, the volume is ${}^{(3)}\mathsf{V} = 2\pi^2$.

$$
\begin{aligned}
\mathcal{H}\Psi &= \left(-\frac{1}{2}\nabla^2 + U\right)\Psi \\
&= \frac{1}{2}\left[\frac{1}{a^3}\left(a\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{\partial^2}{\partial\phi^2}\right) - ka + a^3V(\phi)\right]\Psi \\
&= \frac{1}{2}\mathrm{e}^{-3\alpha}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} - k\mathrm{e}^{4\alpha} + \mathrm{e}^{6\alpha}V(\phi)\right]\Psi = 0\,. \qquad (2.97)
\end{aligned}
$$

In the WKB analysis, assuming a slowly varying potential, suitably approximated by a cosmological constant, so that derivatives with respect to ϕ can be ignored, the first order WKB wave function is

$$
\Psi(a,\phi) \simeq \begin{cases} \dfrac{\mathcal{B}(\phi)}{a\left[a^2V(\phi)-k\right]^{1/4}}\exp\left[\dfrac{\pm\mathrm{i}}{3V(\phi)}\left[a^2V(\phi)-k\right]^{3/2}\right], & a^2V > k\,, \\ \dfrac{\mathcal{C}(\phi)}{a\left[k-a^2V(\phi)\right]^{1/4}}\exp\left[\dfrac{\pm 1}{3V(\phi)}\left[k-a^2V(\phi)\right]^{3/2}\right], & a^2V < k\,. \end{cases} \qquad (2.98)
$$

If V is positive, then oscillatory solutions ($\Psi \sim \mathrm{e}^{\mathrm{i}S}$, where S is a solution of the Hamilton–Jacobi equation, $a^2V \gg |k|$, and $S \simeq \pm a^3\sqrt{V}/3$) will exist for large values of the scale factor, while exponential type solutions will correspond to small values of the scale factor. Note that then

$$
\pi_\alpha = \frac{\partial S}{\partial\alpha} \Longrightarrow \dot{\alpha} \simeq \pm\sqrt{V}\,, \qquad (2.99)
$$

$$
\pi_\phi = \frac{\partial S}{\partial\phi} \Longrightarrow \dot{\phi} \simeq 0\,, \qquad (2.100)
$$

corresponding to an inflationary attractor point.

Note 2.16 It is useful to remark that the minisupermetric (2.93) of the FRW case here is conformal to the two-dimensional Minkowski space in the coordinates (α, ϕ). In a Carter–Penrose conformal diagram (see Fig. 2.3) [20, 38], if we plot $(p-q)$ horizontally and $(p+q)$ vertically, where $\tan p \equiv \alpha + \phi$, and $\tan q \equiv \alpha - \phi$, the boundary consists of points corresponding to:

- past timelike infinity, $i^- = \{(a,\phi)\,|\,a = 0\,,\ \phi \text{ finite}\}$,
- future timelike infinity, $i^+ = \{(a,\phi)\,|\,a = \infty\,,\ \phi \text{ finite}\}$,
- left and right spacelike infinity, $i^0_{\mathrm{L,R}} = \{(a,\phi)\,|\,a = \text{finite}\,,\ \phi = \pm\infty\}$,
- past and future null boundaries, $\mathcal{I}^-_{\mathrm{L,R}} = \{(a,\phi)\,|\,a = 0\,,\ \phi = \pm\infty\}$ and $\mathcal{I}^+_{\mathrm{L,R}} = \{(a,\phi)\,|\,a = \infty\,,\ \phi = \pm\infty\}$.

In each case the subscript L (left) is associated with $\phi \to -\infty$, and the subscript R (right) with $\phi \to +\infty$.

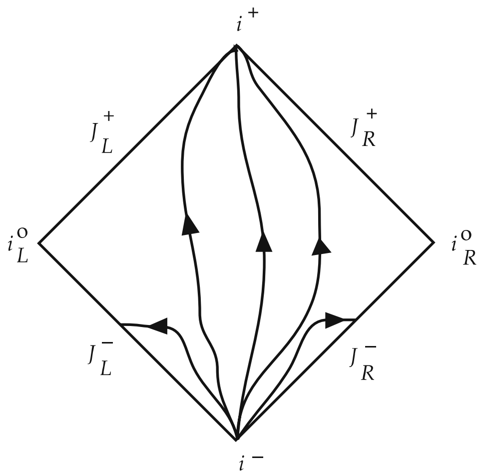

Fig. 2.3 Carter–Penrose conformal diagram for the FRW $k = 1$ minisuperspace [20, 38]

The Hartle–Hawking Boundary Condition

In this section and the next, we shall apply the boundary conditions described in Sects. 2.7.1 and 2.7.2 in the context of minisuperspace quantum cosmology, thereby illustrating by means of simple examples how one can find solutions to the Wheeler–DeWitt equation. Which predictions can thereby be extracted will be the focus of Sect. 2.2 in Vol. II.

The no-boundary wave function for the FRW $k = +1$ model is given by

$$\Psi_{\mathrm{HH}}[a, \phi] = \int^{(a,\phi)} \mathcal{D}a\, \mathcal{D}\phi\, \mathcal{D}\mathcal{N}\, \mathrm{e}^{-I[a,\phi,\mathcal{N}]}\,, \tag{2.101}$$

where

$$I = \frac{1}{2}\int_0^{\tau_{\mathrm{f}}} \mathrm{d}\tau \mathcal{N} \left[\frac{-a}{\mathcal{N}^2}\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 + \frac{a^3}{\mathcal{N}^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 - a + a^3 V \right], \tag{2.102}$$

the integral being taken over a class of paths which match the values $a(\tau_{\mathrm{f}}) = a$, $\phi(\tau_{\mathrm{f}}) = \phi$ on the final surface. The origin of the Euclidean time coordinate τ has been chosen to be zero, so that a saddle-point approximation to the path integral can be made.

If we approach the task of solving the Wheeler–DeWitt equation directly, we are required to extract from the Hartle–Hawking proposal the boundary conditions for Ψ on minisuperspace. Following [20, 96, 30, 38], $\Psi = 1$ when $a \to 0$ with ϕ regular, and also $\Psi = 1$ along the past null boundaries. As Ψ must be regular when $a \to 0$, it follows that Ψ must be independent of ϕ in this limit (i.e., $\partial\Psi/\partial\phi \simeq 0$), so we can ignore the divergence of the a^{-3} factor (see, e.g., [38]).

- For $a^2V \ll 1$, $V \simeq 0$, valid for arbitrarily large $|\phi|$, the Wheeler–DeWitt equation becomes a Bessel equation in the variable $a^2/2$. Thus,

$$\Psi \simeq I_0\left(\frac{1}{2}a^2\right), \tag{2.103}$$

where

$$I_0(z) \equiv \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$$

is the zero-order modified Bessel function.

- For large a (with $a^2V \ll 1$),

$$\Psi \sim \frac{1}{\sqrt{\pi}\,a}\exp\left(\frac{1}{2}a^2\right)\left[1+\mathcal{O}(a^{-2})\right], \tag{2.104}$$

the wave function is of exponential type. It agrees with the WKB approximation in the limit $a^2V \ll 1$ for large a, namely the $(-)$ solution with normalisation

$$\mathcal{C}(\phi) = \frac{1}{\sqrt{\pi}}\exp\left[\frac{1}{3V(\phi)}\right].$$

- For $V(\phi) \gg 1$, the ϕ dependence is *not* significant, and we continue to ignore the ϕ derivative. Further, the spatial curvature is now negligible, and we obtain

$$\Psi \simeq c(\phi)J_0\left(\frac{1}{3}a^3\sqrt{V}\right), \tag{2.105}$$

where

$$J_0(z) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$$

is the zero-order ordinary Bessel function.

- For large a, however,

$$\Psi \sim \frac{c}{\sqrt{2\pi S}}\cos\left(S-\frac{\pi}{4}\right), \tag{2.106}$$

where[35] $S = a^3\sqrt{V}/3$.

---

[35] It should be noted [38] that (2.106) is a *superposition* of the two oscillatory WKB modes for large S (whence decoherence effects should apply [41]). From the WKB connection formula which

If instead we determine Ψ through a saddle-point approximation to the path integral, this leads to a no-boundary wave function

$$\Psi_{\rm HH}(a,\phi) \simeq \begin{cases} \exp\left[\frac{1}{3V(\phi)}\right]\exp\left[\frac{-1}{3V(\phi)}\left[1-a^2V(\phi)\right]^{3/2}\right], & a^2V<1\,, \\ \exp\left[\frac{1}{3V(\phi)}\right]\cos\left[\frac{1}{3V(\phi)}\left[a^2V(\phi)-1\right]^{3/2}-\frac{\pi}{4}\right], & a^2V>1\,, \end{cases} \tag{2.107}$$

where the latter is a superposition of the two WKB components, i.e.,

$$\Psi_{HH} = \Psi_- + \Psi_+\,, \tag{2.108}$$

$$\Psi_\pm \sim \exp\left[\frac{1}{3V(\phi)}\right]\exp\left\{\pm{\rm i}\left[\frac{1}{3V(\phi)}\left[a^2V(\phi)-1\right]^{3/2}-\frac{\pi}{4}\right]\right\}. \tag{2.109}$$

The Vilenkin Proposal

The Vilenkin tunneling proposal within minisuperspace is of considerable interest [31, 32, 35–37]. As the fellow explorer will recall, it involves the notion of minisuperspace boundary and ingoing and outgoing modes, which have a clear definition in this context. Moreover, each oscillatory WKB mode $\Psi \sim {\rm e}^{{\rm i}S_n}$ has a current associated with it, and the notions of ingoing or outgoing can be made through the direction of ∇S_n with respect to a given surface.

In terms of Carter–Penrose diagrams, the following can be extracted for the tunneling condition:

- All surfaces $\mathcal{I}^{\pm}_{\rm L,R}$ and the points $i^0_{\rm L,R}$ and i^+ will be part of the singular boundary.
- The point i^- ($a \to 0$, $\alpha \to -\infty$, ϕ finite) constitutes the only point of the non-singular boundary.

The oscillatory WKB region is bounded by $i^0_{\rm R}$, $\mathcal{I}^+_{\rm R}$, i^+, $i^0_{\rm L}$, and $\mathcal{I}^-_{\rm L}$. In this region, solutions are obviously of the form ${\rm e}^{{\rm i}S_n}$, S_n being a solution to the Hamilton–Jacobi equation [20, 38]

$$-\left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\phi}\right)^2 + U(\alpha,\phi) = 0\,, \tag{2.110}$$

where $U(\alpha,\phi) \equiv {\rm e}^{4\alpha}\left[{\rm e}^{2\alpha}V(\phi)-1\right]$. From (2.110), it follows that

---

matches the (−) solution in the oscillatory region with (2.106),

$$c = \sqrt{\frac{2\pi}{3}}\,\mathcal{C} = \sqrt{\frac{2}{3}}\exp\left[\frac{1}{3V(\phi)}\right].$$

$$\frac{d\alpha}{2S_{,\alpha}} = \frac{d\phi}{2S_{,\phi}} = \frac{dS}{2U} = \frac{dS_{,\alpha}}{U_{,\alpha}} = \frac{dS_{,\phi}}{U_{,\phi}} , \tag{2.111}$$

implying that each $S(\alpha, \phi)$ describes a congruence of classical paths with

$$\frac{d\phi}{d\alpha} = -\frac{S_{,\phi}}{S_{,\alpha}} = \frac{\mp S_{,\phi}}{\sqrt{S_{,\phi}^2 + U}} . \tag{2.112}$$

For $U > 0$, as it is in the oscillatory region if we take $V(\phi) > 0$, these integral curves have $|d\phi/d\alpha| < 1$, with an endpoint at i^+. From $\pi_\alpha \equiv -e^{3\alpha}\dot{\alpha}/\mathcal{N} = S_{,\alpha}$, these WKB modes are associated with expanding universes ($\dot{\alpha} > 0$) for $\pi_\alpha < 0$, or contracting universes for $\pi_\alpha > 0$, with $\mathcal{N} > 0$. As a result:

- All paths originate at finite values of α.
- The contracting solutions will extend from i^+ into the interior of the minisuperspace, and the ingoing modes are excluded through the tunneling boundary condition.
- Only the expanding solutions (outgoing at i^+) are allowed, i.e., modes like

$$\Psi_V \sim \exp\left\{ \frac{-i}{3V(\phi)} \left[e^{2\alpha} V(\phi) - 1 \right]^{3/2} \right\} . \tag{2.113}$$

- The wave function is *complex* in the oscillatory region (the no-boundary wave function is *real*).

Summary and Review. Before proceeding with a (very) brief overview of SUSY and SUGRA in the next chapter, the resolute explorer is invited to review the main elements of this chapter. This will be done through a series of questions, rather than merely repeating the contents of the above discussion. It will be an opportunity to revise and consolidate understanding. Note that the relevant sections are numbered between brackets to assist the reader in this process:

1. Why is QC well-motivated for research study (Sect. 2.1)?
2. Describe the 'historical' sequence of events and *who* contributed *what* within QC (Sect. 2.1).
3. How can the Hamiltonian (i.e., the Wheeler–DeWitt) and momentum constraint equations be extracted from a 3 + 1 description of spacetime (Sects. 2.2 and 2.5)?
4. What are superspace and minisuperspace (Sects. 2.5 and 2.7)?
5. Why is it so important to discuss the corresponding algebra of constraints? What physical information can be retrieved, in particularly, concerning the geometry of the spacetime (Sect. 2.6)?

6. What are the main features of the ADM canonical formalism (Sects. 2.3 and 2.4)?
7. What are the main differences between the ADM and the Dirac quantization schemes for gravity (Sect. 2.4)?
8. What are consistent ‘coordinate conditions’ (Sect. 2.3.2)?
9. How did the ADM canonical formalism provide a *glimpse* of SQC (Sect. 2.4.1)?
10. Indicate the assumptions and properties of the no-boundary and tunneling wave functions for the universe (Sects. 2.7 and 2.8).

Problems

2.1 Canonical Form of General Relativity, Vierbein, and Lorentz Constraint
From the Einstein–Hilbert action expressed in terms of a vierbein (tetrad) as basic fields, retrieve the (local) Lorentz invariance (see [76]).

2.2 FRW and Bianchi IX from the ADM and Dirac Perspectives
Investigate the closed FRW with dust and the Bianchi IX *symmetric* case from the ADM and Dirac perspectives (see [43]).

2.3 Momentum Constraints and Spatial Invariance
Show that the momentum constraints in the form of (2.71) imply that the wave function is the same for configurations (h_{ij}, ϕ) that are related by coordinate transformations in the 3-surfaces.

2.4 Geodesics from Quantum Cosmology
From a quantum cosmological perspective, retrieve a geodesic equation assigned to Einstein’s equations.

2.5 Killing Vectors and Constants of Motion in Quantum Cosmology
Investigate how Killing vectors can identify constants of motion within the mathematical structure of minisuperspace (see [43]).

2.6 Minisuperspace Metrics for FRW and Bianchi IX
Find the minisuperspace metrics for FRW, diagonal Bianchi IX, symmetric Bianchi IX, and generic Bianchi within the Misner–Ryan parametrization (see [43]).

2.7 WKB and Classical Limit
Within the context of a WKB analysis, investigate the closed FRW model with a scalar field. Conclude that classical behaviour is associated with oscillatory stages, whereas classically inaccessible epochs are related to e^I terms.

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Chapter 3
A Summary of Supersymmetry and Supergravity

Supersymmetry (SUSY) is an attractive concept whose basic feature is a transformation which relates bosons to fermions and vice-versa [1–13]. Its promotion to a gauge symmetry has resulted in an elegant field theory called supergravity (SUGRA) [14–20]. One of its most significant features is that the presence of *local* SUSY naturally implies spacetime to be curved. Hence, gravity must *necessarily* be present. SUGRA constitutes an extension of the general relativity theory of gravity. Perhaps, then, in this context, we could suggest that SUSY becomes meaningful *if* operating within a SUGRA setting.

$N = 1$ SUGRA is the simplest theory with *one* real massless gravitino ($N = 0$ corresponds to general relativity). $N = 2$ SUGRA realises Einstein's dream of unifying gravity with electromagnetism. This theory contains two gravitinos besides the gravitational and Maxwell fields. It was in this theory that finite probabilities for loop diagrams with gravitons were first obtained. In particular, the photon–photon scattering process, which is divergent in an Einstein–Maxwell theory, was shown to be *finite* when $N = 2$ SUGRA was considered [21, 22].

Note 3.1 An unambiguous proof for the complete finiteness of a SUGRA theory remains elusive, although $N = 8$ SUGRA has come tantalizing close, being shown to be finite up to the 3-loop range. (It was a remarkable feat to achieve this!) However, this was not enough, and full finiteness within a SUSY setting has only been provided by inviting string dynamics into the programme.

The fact that string theory relates to SUGRA (in particular to $N = 1$, 11-dimensional SUGRA within M-theory) adds to the hope that $N = 8$ SUGRA may eventually be fully finite after all. But we still lack the ingredients to prove that.

The issue of whether SUGRA is fully finite is of great relevance, and we will outline a few arguments in Chap. 4 of Vol. II. Moreover, it has become a sensitive issue involving canonical quantization methods, as conflicting views have been expressed [23–29].

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At large scales, SUGRA makes the same predictions for classical tests as general relativity. But at small (microscopic) scales, SUGRA quantum effects may instead bring a cancellation of infinities otherwise present in several purely bosonic approaches to a quantum gravity theory. In particular, supersymmetry plays an important role when dealing with ultraviolet divergences in quantum cosmology and gravity [30–34, 27, 35], and removing Planckian masses induced by wormholes [26]. For these and other reasons (e.g., being 'so beautiful it must bear some truth' [8, 10, 20]), some researchers hope that nature has reserved a rightful place for SUSY and SUGRA [17]. Therefore, it would probably be adequate for the purpose of studying the very early universe to consider scenarios where all bosonic and fermionic matter fields were present on an equal footing.

3.1 Why SUSY?

Since there is no clear and unambiguous experimental evidence for (SUSY) [9, 12], the question in the section title is quite relevant. In fact, it can perhaps be rephrased as follows: Is there any need for SUSY?

Before proceeding into more technical grounds, from an aesthetical perspective, a theory which is supersymmetric is highly symmetric. It would *unify* fermions (constitutents of matter) with bosons (carriers of force), which in a curved spacetime leads to SUGRA. And the relevance of SUGRA extends beyond this fact. It is an ambitious and elegant framework, as it attempts to contribute (currently, within superstring and M-theory) [36–43] to a complete unification of the gravitational field with other interactions. However, this involves peculiar features. On the one hand, general relativity posits the gravitational field, not as an interaction, but as global property of space and time as a whole. On the other hand, the strong, weak, and electromagnetic interactions constitute quantum field theories, where *local gauge invariance* is required. From Noether's theorem [44], in a theory with invariance under a global symmetry, i.e., not dependent on spacetime coordinates, there is an associated conserved current and charge. But for local gauge invariance, under transformations dependent on spacetime coordinates, an additional field must be introduced, namely the gauge potential, generating forces between the charged particles. In brief, gauging determines the appearance of interactions and therefore of forces. It is in this context that local gauged supersymmetry implies, in particular, gravity. Of course, it is also well known that, so far, supersymmetric theories have not yet made contact with physical reality and become an underlying fundamental theory for nature.

But SUSY surely seems both powerful and beautiful, assisting in overcoming many obstacles facing theoretical physics. Concerning more technical endorsements, there are many routes to introduce the motivation for SUSY, and in this section we summarize a well-known argument (see [12] for more details) that is perhaps the closest to upcoming results from observations at the LHC.

Paradoxically, the main reason for SUSY might be the fact that the standard model of particle physics is so accurate concerning experimental data. Indeed, it turns out to be an extremely efficient framework, although the whole structure is based on the existence of the elusive Higgs particle (hopefully, the LHC will soon detect it), with an *appropriate* mass. And this is the key problem. The computed value of the mass is quite sensitive, in a rather *critical* manner, to quantum corrections.

The example in [12] proceeds by calculating the largest contribution to the Higgs mass through the *top quark* (the attentive explorer will notice that this is a fermion term) coupling to the Higgs. The crucial feature is that there are quadratically and logarithmically divergent corrections. Besides the obvious but unpleasant fine-tuning procedure, an upper cutoff is needed at about 1 TeV. The standard model is reasonable as long as this cutoff is present, and beyond it, for stability requirements, *something new* ought to be present. But what could that be?

It turns out that a simple procedure can stabilize the computed mass of the Higgs by cancelling the divergent corrections. The idea is to include some new scalar particles, i.e., bosonic terms, the reader will notice! More precisely, with a suitable number of *new* scalar particles, and suitable choices for their masses and coupling parameters, the quadratic and logarithmic divergences can be duly cancelled. Hence, the recipe seems to be to add bosons (fermions) when the divergences and instabilities are caused by fermions (bosons). And that is what SUSY brings about.

Another way of exemplifying the benefits is via the *hierarchy problem* [9]. The energy gap between GUTs (grand unification theories) and W boson masses is not generally *stable* in perturbation theory (unless repeated fine-tuning operations are employed at each order). In the presence of SUSY, this is avoided and the gap becomes stabilized.

In fact, SUSY proposes that for every ordinary particle there exists a *superpartner*, hence specifically relating particles with *adjacent* spins. Any such fermions and bosons can be manifestations of a single *superparticle*, like a very peculiar ‘arrow’ in a suitable auxiliary space. Supersymmetry transformations result in a change in the ‘orientation’ of a particle. As a consequence, the generic feature (in terms of quite powerful *non-renormalization* theorems) is that radiative or loop corrections tend to be less important in supersymmetric theories, due to cancellations between fermion loops and boson loops. Quantities that are small or vanish classically (i.e., at tree level) will remain so once quantum corrections (loops) are inserted. It all sounds too good to be true. However, SUSY has not yet been detected within the range of currently accessible experimental energies. This means that SUSY is broken in that range, but if the energy goes high enough, SUSY may become accessible to observation.

Note 3.2 It would also be an oversight not to provide an account of how SUSY actually emerged during the 1970s. The official line goes as follows. In the search for unified field theories, looking for wider gauge groups that could encompass all the known interactions and extend beyond the Poincaré symmetry, a remarkable result was presented: the Coleman–Mandula no-go theorem [45, 46, 10, 13, 47]. In essence, the only symmetry of the scattering matrix (S-matrix) that included the Poincaré symmetry was in fact the product of the Poincaré symmetry with some internal symmetry group. (For example, the electrical charge operator has eigenvalues, or quantum numbers, that are rotationally and translational invariant. They *cannot* relate states with different values of the mass or spin. Irreducible multiplets cannot contain particles of different mass or spin.) Any other Lorentz-like generators would lead to inconsistencies.

However, a *lateral* route of exploration was devised by Golfand, Likhtman, Wess, Zumino, Akulov, and Volkov [48–52, 10, 53, 54]. The original no-go theorem invoked symmetry transformations of a Lie group nature, with *real* parameters, and hence commutation relations for their generators. But different spins in the same multiplet are allowed if an extension of the Poincaré algebra to a graded Lie algebra is made (with anticommutators and spinor generators), bringing about a symmetry intertwining fermions and bosons, not generating Lie groups, and keeping within the Coleman–Mandula requirements.

So why is this relevant? Well, it was proved soon afterwards [46, 10] that SUSY is *the only possible extension* of the Poincaré algebra that is consistent with an extension of the Coleman–Mandula no-go theorem. In this way, SUSY revealed itself to be a highly relevant element in any route of exploration into the undiscovered realm of a fully unified theory of interactions. To be more precise, the no-go theorem implies that supersymmetry is the *only* possible ingredient for unification within a relativistic quantum theory (with superstrings and M-theory being the currently most promising line). Spacetime and internal symmetries can only be related by (fermionic) symmetry operators Q of spin 1/2 (see Exercise 3.1) with

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\ , \qquad Q\,|\text{boson}\rangle = |\text{fermion}\rangle\ . \tag{3.1}$$

However, an even more attractive feature from SUSY should be mentioned.

A repeated supersymmetry application moves a particle from one position to another in spacetime (see Sect. 3.2 and Exercise 3.2). On the one hand, this means that the commutator of two SUSY transformations would be another symmetry transformation and the algebra will close in this manner. And on the other hand, since local Poincaré invariance is the symmetry inducing general relativity, a connection between supersymmetry, gravitation, and the structure of spacetime, can be foreseen. When SUSY is promoted to *local* invariance, *new* gauge fields and hence *new* forces arise. These will be the spin 2 graviton field and a new spin 3/2 field, the gravitino, allowing the construction of the new gravitational theory known as SUGRA [1, 8, 11, 20, 47]. It then provides corrections to the general theory of relativity at the quantum level. There is an additional contribution from the exchange of spin 3/2 gravitinos. Long-range interactions are unchanged, but *new* effects are predicted at the *microscopic* scale (see Sect. 3.4.3).

3.2 SUSY Algebra and Transformations

The aim of this section is to introduce the main features of SUSY, assuming acquaintance with some of the essentials of spinors (and fermions) and their transformations (if necessary, see Sect. A.3). We start by presenting the main features of a SUSY algebra and the important physical consequences. We then discuss how fields are affected by a SUSY transformation, in order to investigate SUSY invariance.

3.2.1 SUSY Algebra

SUSY transforms bosons into fermions and vice-versa (see Exercise 3.3). Hence, this operation must be intertwined with the Poincaré algebra by suitable generators. It can be shown that (see Appendix A for conventions and notation, relevant spinor quantities and their definitions, and in particular the 2-component and 4-component representations):

- SUSY operators transform either as unprimed $(1/2, 0)$ spinors Q_A or as primed $(0, 1/2)$ spinors $\overline{Q}_{A'}$ under the Lorentz group, where $A = \{0, 1\}$, $A' = \{0', 1'\}$ (see Exercise 3.1).
- SUSY operators commute with translations and intertwine with Lorentz transformations according to:

$$[\mathcal{P}_\mu, Q_A] = 0\,, \tag{3.2}$$

$$[\mathcal{P}_\mu, \overline{Q}_{A'}] = 0\,, \tag{3.3}$$

$$[\mathcal{L}_{\mu\nu}, Q_A] = -\mathrm{i}(\sigma_{\mu\nu})_A{}^B Q_B\,, \tag{3.4}$$

$$[\mathcal{L}_{\mu\nu}, \overline{Q}_{A'}] = -\mathrm{i}(\overline{\sigma}_{\mu\nu})^{A'}{}_{B'}\overline{Q}^{B'}\,. \tag{3.5}$$

In addition (see Exercise 3.2), the SUSY algebra is not exausted by (3.2), (3.3), (3.4), and (3.5). Spinors anticommute, and the most significant feature of SUSY derives from this feature. Since the Q_A transform in the $(1/2, 0)$ representation and the $\overline{Q}_{A'}$ in the $(0, 1/2)$, the anticommutator of Q_A and $\overline{Q}_{A'}$ must transform as $(1/2, 1/2)$, i.e., as a 4-vector! The following algebra is then retrieved

$$\{Q_A, \overline{Q}_{A'}\} = 2\sigma^{\mu}_{AA'}\mathcal{P}_{\mu}\,, \tag{3.6}$$

$$\{Q_A, Q_B\} = 0\,, \tag{3.7}$$

$$\{\overline{Q}_{A'}, \overline{Q}_{B'}\} = 0\,. \tag{3.8}$$

Note 3.3 The algebra specified by (3.6), (3.7), and (3.8) comprises a single set of operators $(Q_A, \overline{Q}_{A'})$, also known in the literature as (super)charges. But why do we not have more? The situation in (3.6), (3.7), and (3.8) is referred to as $N = 1$ SUSY, but we could instead have sets $Q^I{}_A, \overline{Q}^J_{A'}, I, J = 1, \ldots, N$ for N-extended SUSY, with I, J labeling some internal symmetry to which the SUSY operators belong. (But can N be increased without bound? Increasing N implies including particles of increasing spin, and there seem to be no consistent quantum field theories with spins larger than one (without gravity) or larger than two (with gravity), leading therefore to $N \leq 4$, or $N \leq 8$.) After diagonalizing and rescaling, we then have

$$\{Q^I_A, \overline{Q}^J_{A'}\} = 2\sigma^{\mu}_{AA'}\mathcal{P}_{\mu}\delta^{IJ}\,, \tag{3.9}$$

$$\{Q^I_A, Q^J_B\} = \varepsilon_{AB}Z^{IJ}\,, \tag{3.10}$$

$$\{\overline{Q}^I_{A'}, \overline{Q}^J_{B'}\} = \varepsilon_{A'B'}(Z^{IJ})^*\,, \tag{3.11}$$

where Z^{IJ} constitutes a linear combination of the internal symmetry generators. (By diagonalising an a priori arbitrary symmetric matrix and rescaling the Q and $\overline{Q}$, one can always obtain δ^{IJ}. Further, as $\overline{Q}$ is the Hermitian conjugate of Q, positivity of the Hilbert space excludes zero eigenvalues for this matrix.) Note that the $Z^{IJ} = -Z^{JI}$ constitute the *central charges*, i.e., they commute with *all* generators of the full algebra. The simplest algebra (unextended SUSY) has $N = 1$, i.e., there are no indices I, J, no central charges, and the internal group is $U(1)$ or R-symmetry [12]. For $N > 1$, we have *extended* supersymmetry, e.g., $N = 2$ implies just one central charge $Z \equiv Z^{12}$.

From the above, the following properties can be retrieved:

- All particles belonging to an irreducible representation of SUSY, i.e., within one supermultiplet, have the same mass (a variant of O'Raifeartaigh's theorem [1, 5, 8, 10, 12]). This is obvious since $\mathcal{P}^2$ commutes with all generators of the SUSY algebra, i.e., it is still a Casimir operator.
- In a supersymmetric theory the energy $\mathcal{P}_0$ is always positive. To see this, let $|\Psi\rangle$ be any state. Then by the positivity of the Hilbert space,

$$
\begin{aligned}
0 &\leq \left\| Q^I_A |\Psi\rangle \right\|^2 + \left\| (Q^I_A)^\dagger |\Psi\rangle \right\|^2 \\
&= \langle\Psi| \left[(Q^I_A)^\dagger Q^I_A + Q^I_A (Q^I_A)^\dagger |\Psi\rangle \right] \\
&= \langle\Psi| \{Q^I_A, \overline{Q}^I_{A'}\} |\Psi\rangle = 2\sigma^\mu_{AA'} \langle\Psi| \mathcal{P}_\mu |\Psi\rangle \ ,
\end{aligned} \tag{3.12}
$$

using $(Q^I_A)^\dagger \equiv \overline{Q}^I_{A'}$, where the dagger denotes Hermitian conjugate (see Appendix A). Summing over A and A' and using Tr $\sigma^\mu = 2\delta^{\mu 0}$, this implies $0 \leq 4 \langle\Psi| \mathcal{P}_0 |\Psi\rangle$ (see Sect. B.2).

- A supermultiplet[1] always contains an equal number of bosonic and fermion degrees of freedom, i.e., effective physical (positive norm) states (see Exercise 3.4).

3.2.2 Supermultiplets

But how can a supermultiplet be consistently constructed? From the SUSY algebra we can construct representations in the form of particle (super)multiplets. Massive particles will be labeled by mass, total spin, and one spin component, e.g., $|m, s, s_z\rangle$, while massless particles will instead be labeled by energy and helicity. A useful feature is that the full SUSY algebra contains the Poincaré algebra as a *sub*algebra. This means that any representation of the full SUSY algebra induces a representation of the Poincaré algebra (generally, reducible[2]). An irreducible representation of the SUSY algebra therefore in general corresponds to several particles. The corresponding states (particles) are related through the application of the Q^I_A and $\overline{Q}^I_{A'}$, and thus have spins differing by 1/2. They constitute a *supermultiplet*.

As an example [55, 12], consider the massless case $\mathcal{P}^2 = 0$. For simplicity, we start with $N = 1$ SUSY (see Exercise 3.5), i.e., no central charges and Z^{IJ} absent. Within the reference frame where[3] $\mathcal{P}_\mu = (E, 0, 0, E)$,

$$
\sigma^\mu P_\mu = \begin{bmatrix} 0 & 0 \\ 0 & -2E \end{bmatrix} \quad \Longrightarrow \quad \{Q_A, \overline{Q}_{A'}\} = \begin{bmatrix} 0 & 0 \\ 0 & -4E \end{bmatrix}_{AA'} . \tag{3.13}
$$

But this means $\{Q_1, \overline{Q}_{1'}\} = 0$, implying $Q_1 = \overline{Q}_{1'} = 0$. We then define a vacuum state $|\mathsf{O}\rangle$ as $|\mathsf{O}\rangle \equiv Q_0 |E, \ell\rangle$, $Q_0 |\mathsf{O}\rangle = 0$. From $\{Q_1, \overline{Q}_{1'}\} = 0$ applied to $|\mathsf{O}\rangle$, this implies that $\overline{Q}_{1'}$ produces states of zero norm. The vacuum state $|\mathsf{O}\rangle$, carries some irreducible representation of the Poincaré algebra, i.e., zero mass but characterized by some helicity ℓ. Since Q_0 lowers the helicity by one half and $\overline{Q}_{0'}$ raises it by one half, the supermultiplet will be formed by

---

[1] A supermultiplet must contain at least one boson and one fermion, whose spins differ by 1/2.

[2] Each *irreducible* representation of the Poincaré algebra corresponds to a particle. More precisely, the Poincaré group particle representations are associated with values of the mass and helicity.

[3] For conventions see Appendix A as well as, e.g., [11, 47, 56] for *equivalent* formulations.

$$\left|\mathsf{O}\right\rangle, \qquad \overline{Q}_{0'}\left|\mathsf{O}\right\rangle \equiv \left|\mathsf{O}+1/2\right\rangle, \tag{3.14}$$

containing only two states $\left|\mathsf{O}\right\rangle$ and $\left|\mathsf{O}+1/2\right\rangle$, denoted by $\left(\mathsf{O},\mathsf{O}+1/2\right)$. We may then proceed accordingly and establish the following multiplets:

- Chiral multiplet. This consists of (0, 1/2), corresponding to a Weyl fermion and a complex scalar.
- Vector multiplet. This consists of (1/2, 1), corresponding to a gauge boson (massless vector) and a Weyl fermion.
- Gravitino multiplet. This contains (1, 3/2) and (−3/2, −1), i.e., a gravitino and a gauge boson.
- Graviton multiplet. This contains (3/2, 2) and (−2, −3/2), corresponding to the graviton and the gravitino.

3.2.3 SUSY Transformations and the Wess–Zumino Model

Although interesting in itself, the reader may be wondering whether this section is relevant for SQC, since we are still presenting SUSY from a *global* perspective, and most of the SQC models will be retrieved from $N = 1$ SUGRA. A suitable answer will only emerge in subsequent chapters, where the breadth of scope of SQC (see Part III) does in fact require the discussion below. Moreover, there will be *formal similarities*, especially with regard to *transformation properties*, for a number of cosmological models. We will see in Chap. 5 how to retrieve SQC with $N = 4$ (*local*) SUSY from $N = 1$ SUGRA. And in Chap. 6, we will see how to use models for SQC built instead with $N = 2$ SUSY, where the contents of this section (and this chapter, as a matter of fact) will be important, because they may assist in subsequent investigation of open issues in SQC. The resilient explorer should bear this in mind when we arrive at that point.

To illustrate our argument, let us outline how to arrive at the Wess–Zumino model [1–3, 6, 10–12, 57–59]. In particular, we will extract the transformation of the fields under SUSY in a way that is independent of the interactions in the model and does *not* require the fields to be *on-shell*.

Let us then take the Lagrangian for a set of scalar fields and a Majorana spinor (see Sect. A.3.2) in the 4-component spinor [6, 47, 56]:

$$\begin{aligned} L &= -\frac{1}{2}\left(\partial\phi_1\right)^2 - \frac{1}{2}\left(\partial\phi_2\right)^2 - \frac{\mathrm{i}}{4}\overline{\psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi \\ &= -\partial^{\mu}\phi^{*}\partial_{\mu}\phi - \frac{\mathrm{i}}{4}\overline{\psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial}_{\mu}\psi\,, \end{aligned} \tag{3.15}$$

where $\overset{\leftrightarrow}{\partial}_{\mu} \equiv \overrightarrow{\partial_{\mu}} - \overleftarrow{\partial_{\mu}}$ and $\phi \equiv \left(\phi_1+\phi_2\right)/\sqrt{2}$. In the Weyl 2-component spinor $\psi \equiv \begin{pmatrix}\psi_A\\ \overline{\psi}^{A'}\end{pmatrix}$ notation,

$$L \simeq \partial^\mu \phi^* \partial_\mu \phi + \frac{\mathrm{i}}{2} \overline{\psi}^{A'} \sigma^\mu_{AA'} \overleftrightarrow{\partial_\mu} \psi^A \longrightarrow \partial^\mu \phi^* \partial_\mu \phi + \mathrm{i} \psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi \,, \tag{3.16}$$

with the latter expression being equal up to addition of a total derivative. For SUSY transformations $\phi \to \phi + \delta\phi$, $\psi \to \psi + \delta\psi$, we can take for the scalar $\delta\phi = \varepsilon^A \psi_A$, $\delta\phi^* = \varepsilon^\dagger_{A'} \psi^{\dagger A'}$, ε^A an infinitesimal spinor (using some dimensional guesswork[4]), to act on the kinetic term $\partial^\mu \phi^* \partial_\mu \phi$. For the fermions, requiring linearity in ϕ and ε, and noticing that the scalar sector of the Lagrangian has two derivatives while the fermionic sector only has $\mathrm{i}\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi$, the SUSY transformation has to have one derivative in ϕ to ensure cancellation. Hence, we find $\delta\psi_A = \mathrm{i} \left(\sigma^\nu \varepsilon^\dagger\right)_A \partial^\nu \phi$ and $\delta\psi^\dagger_{A'} = -\mathrm{i} \left(\varepsilon\sigma^\nu\right)_{A'} \partial^\nu \phi^*$, cancelling up to a total derivative.

This then also confirms that the commutator of two SUSY transformations is a symmetry transformation. More precisely, it can be shown that (see Exercise 3.2 for additional details)

$$[\delta_{\varepsilon 1}, \delta_{\varepsilon 2}]\, \phi = -\mathrm{i} \left(\varepsilon_1 \sigma^\nu \varepsilon_2^\dagger - \varepsilon_2 \sigma^\nu \varepsilon_1^\dagger \right) \partial_\nu \phi \,, \tag{3.17}$$

$$\begin{aligned} [\delta_{\varepsilon 1}, \delta_{\varepsilon 2}]\, \psi_A = &-\mathrm{i} \left(\varepsilon_1 \sigma^\nu \varepsilon_2^\dagger - \varepsilon_2 \sigma^\nu \varepsilon_1^\dagger \right) \partial_\nu \psi_A \\ &+\mathrm{i} \left(\varepsilon_{1A} \varepsilon_2^\dagger \overline{\sigma}^\nu \partial_\nu \psi - \varepsilon_{2A} \varepsilon_1^\dagger \overline{\sigma}^\nu \partial_\nu \psi \right) , \end{aligned} \tag{3.18}$$

with the last term vanishing *on-shell* (from the equation of motion $\overline{\sigma}^\mu \partial_\mu \psi = 0$). Hence, the commutator of two SUSY transformations induces a spacetime translation.

Notice then the following count of degrees of freedom (d.o.f.) [12]:

Degrees of freedom	Off-shell	On-shell
ϕ, ϕ^*	2	2
$\psi_A, \psi^\dagger_{A'}$	4	2

The reduction for the fermions is due to the use of the equations of motion.

To make SUSY manifest *off-shell*, one must add *auxiliary* fields (with dimension 2), which have *no* kinetic term. Using complex scalars with $f \equiv (f_1 + \mathrm{i} f_2)/\sqrt{2}$ and SUSY transformations

$$\delta f = \mathrm{i}\varepsilon^\dagger \overline{\sigma}^\nu \partial_\nu \psi \,, \qquad \delta f^* = -\mathrm{i} \partial_\nu \psi^\dagger \overline{\sigma}^\nu \varepsilon \,, \tag{3.19}$$

and adding

[4] With $c = 1 = \hbar$, the dimension of the Lagrangian in units of inverse length is 4. Then ϕ and ψ have dimensions 1 and 3/2, respectively. Hence, for $\delta\phi$ we find ε with mass dimension $-1/2$. For Q or $\overline{Q}$, the mass dimension is $1/2$, rendering εQ dimensionless.

$$\delta\psi_A = \mathrm{i}\left(\sigma^\nu\varepsilon^\dagger\right)_A \partial^\nu\phi + \varepsilon_A f\,, \tag{3.20}$$

$$\delta\psi^\dagger_{A'} = -\mathrm{i}\left(\varepsilon\sigma^\nu\right)_{A'} \partial^\nu\phi^* + \varepsilon^\dagger_{A'} f^*\,, \tag{3.21}$$

then including $f^* f$ in the action makes it invariant off-shell as well, with the algebra closing. Note that the auxiliary fields have no effective degrees of freedom on-shell, only two degrees of freedom off-shell. The action retrieved as

$$L^{\mathrm{WZ}} \simeq -\partial^\mu\phi^{\mathsf{I}*}\partial_\mu\phi_{\mathsf{I}} - \mathrm{i}\psi^{\mathsf{I}\dagger}\overline{\sigma}^\mu\partial_\mu\psi_{\mathsf{I}} + f^{*\mathsf{I}} f_{\mathsf{I}} \tag{3.22}$$

describes the well known Wess–Zumino model, with the additional arbitrary (group) index I labeling *different* chiral multiplets (see Notes 3.8 and 3.9).

Interaction terms can be constructed. An illustrative case is the simple mass term

$$L_{\mathrm{int}} = m\left[-\frac{1}{2}\left(\psi^A\psi_A + \overline{\psi}_{A'}\overline{\psi}^{A'}\right) + \phi_1 f_1 + \phi_2 f_2\right]. \tag{3.23}$$

The auxiliary fields can be eliminated through their equations of motion

$$f_1 = -m\phi_1\,, \quad f_2 = -m\phi_2\,, \tag{3.24}$$

obtained from

$$L^{\mathrm{WZ}} \simeq -\partial^\mu\phi^*\partial_\mu\phi - \mathrm{i}\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + f^* f + L_{\mathrm{int}}\,, \tag{3.25}$$

which, substituting back into (3.25), yields

$$L^{\mathrm{WZ}} \simeq -\partial^\mu\phi^*\partial_\mu\phi - \mathrm{i}\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}\left(\psi^A\psi_A - \overline{\psi}_{A'}\overline{\psi}^{A'}\right) - m^2\phi^*\phi\,. \tag{3.26}$$

More generally [1, 7, 8, 10], as dimensional analysis will confirm, an interaction term would have the form

$$L_{\mathrm{int}} \sim -\frac{1}{2}\mathsf{W}^{\mathsf{JK}}\psi_{\mathsf{J}}\psi_{\mathsf{J}} + \mathsf{W}^{\mathsf{J}} f_{\mathsf{J}} + \text{h.c.}\,, \tag{3.27}$$

where

$$\mathsf{W}^{\mathsf{J}} = \frac{\partial\mathsf{W}}{\partial\phi_{\mathsf{J}}}\,, \qquad \mathsf{W}^{\mathsf{JK}} = \frac{\partial^2\mathsf{W}}{\partial\phi_{\mathsf{J}}\partial\phi_{\mathsf{K}}}\,, \tag{3.28}$$

with W^{JK} a symmetric linear function in ϕ and ϕ^*, under $\mathsf{K} \leftrightarrow \mathsf{J}$. Invariance under SUSY requires that W^{JK} be analytic (W^{JK} holomorphic). W is called the *superpotential*. *No* particular form for W is stipulated. It is basically only required to be holomorphic in ϕ. Renormalizability indicates

$$\mathsf{W} \sim E^{\mathsf{J}}\phi_{\mathsf{J}} + \frac{1}{2}M^{\mathsf{JK}}\phi_{\mathsf{J}}\phi_{\mathsf{K}} + \frac{1}{6}N^{\mathsf{JKN}}\phi_{\mathsf{J}}\phi_{\mathsf{K}}\phi_{\mathsf{N}}\,, \tag{3.29}$$

where E, M, N are coupling constants.

Note 3.4 It should be remarked that (see also (3.24)), if we substituted the solutions for the auxiliary fields in the SUSY transformations of the fermion, they would be different for each choice of the superpotential! However, this problem does *not* arise when working off-shell. Hence, using the solutions $f_{\mathsf{J}} = -\mathsf{W}^*_{\mathsf{J}}$ and $f^*_{\mathsf{J}} = -\mathsf{W}_{\mathsf{J}}$, the Lagrangian becomes

$$\begin{aligned} L \simeq & -\partial^{\mu}\phi^{\mathsf{J}*}\partial_{\mu}\phi_{\mathsf{J}} - \mathrm{i}\psi^{\mathsf{J}\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\psi_{\mathsf{J}} - \mathsf{W}^{*\mathsf{J}}\mathsf{W}_{\mathsf{J}} \\ & -\frac{1}{2}\left(\mathsf{W}^{\mathsf{JK}}\psi_{\mathsf{J}}\psi_{\mathsf{K}} + \mathsf{W}^{*\mathsf{JK}}\psi^{\dagger}_{\mathsf{J}}\psi^{\dagger}_{\mathsf{K}}\right) \end{aligned} \tag{3.30}$$

and for $E = 0$ to avoid SUSY breaking (see Sect. 3.4 of Vol. II), with the following scalar potential

$$V(\phi, \phi^*) = \mathsf{W}^{*\mathsf{J}}\mathsf{W}_{\mathsf{J}} = f_{\mathsf{J}} f^{*\mathsf{J}}\,, \tag{3.31}$$

and using (3.29) explicitly in the computations, the full Lagrangian for the interacting Wess–Zumino model is

$$\begin{aligned} L^{\mathrm{WZ}} \simeq & -\partial^{\mu}\phi^{\mathsf{J}*}\partial_{\mu}\phi_{\mathsf{J}} - \mathrm{i}\psi^{\mathsf{J}\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\psi_{\mathsf{J}} - V(\phi, \phi^*) \\ & -\frac{1}{2}\left(M^{\mathsf{JK}}\psi_{\mathsf{J}}\psi_{\mathsf{K}} + M^{*\mathsf{JK}}\psi^{\dagger}_{\mathsf{J}}\psi^{\dagger}_{\mathsf{K}}\right) \\ & -\frac{1}{2}N^{\mathsf{JKN}}\phi_{\mathsf{J}}\psi_{\mathsf{K}}\psi_{\mathsf{N}} - \frac{1}{2}N^{*\mathsf{JKN}}\phi^*_{\mathsf{J}}\psi^{\dagger}_{\mathsf{K}}\psi^{\dagger}_{\mathsf{N}}\,. \end{aligned} \tag{3.32}$$

3.3 Superspace and Superfields

Although not widely explored in SQC (but see Chap. 6 of Vol. I and Chap. 8 of Vol. II), a powerful way of recovering supersymmetric quantum theories is to establish the specific representations of the SUSY algebra on fields, and the way these transform through (another!) superspace. This is a *different* space from the one introduced in Chap. 2. The idea is that, since the momentum operator $\mathcal{P}_{\mu} \sim -\mathrm{i}\partial_{\mu}$ is the generator of translations in spacetime coordinates x^{μ}, we would like to be able to define (Grassmannian) coordinates θ_A and $\overline{\theta}_{A'}$, where the (Grassmannian) SUSY generators[5] would act like differential operators as $Q_A \sim \partial_{\theta_A}$ and $\overline{Q}_{A'} \sim \partial_{\overline{\theta}_{A'}}$.

[5] We follow [47] in using Q, $\overline{Q}$ for either differential operators or group generators.

In very simple terms, superspace would enlarge the usual spacetime manifold (where coordinates are x^μ) by adding two plus two anticommuting Grassmannian coordinates (constant, i.e., x^μ-independent, spinors) θ_A and $\overline{\theta}_{A'}$. The coordinates on superspace[6] are therefore $(x^\mu, \theta_A, \overline{\theta}_{A'})$ [1, 7, 12, 20]. The reader may be wondering what the actual physical meaning of such a space might be. We will provide some elements of an answer in the next chapter (but a complete appraisal requires further elaboration [60, 61]). Here we will simply exemplify its usefulness and efficiency within SUSY (and subsequently, SUGRA).

For the moment, let us proceed by checking how the SUSY operators Q_A and $\overline{Q}_{A'}$ may eventually act as generators of *super* translations (see [47] and, e.g., [11] for an equivalent description). It is a first step toward establishing the way SUSY generators could act as $Q_A \sim \partial_{\theta_A}$ and $\overline{Q}_{A'} \sim \partial_{\overline{\theta}_{A'}}$. A finite supertranslation $\mathcal{P}^{\mathrm{s}}$ is then given by

$$\mathcal{P}^{\mathrm{s}}(a^\mu, \varepsilon_A, \overline{\varepsilon}_{A'}) \equiv \exp\left(-\mathrm{i}a^\mu \mathcal{P}_\mu + \mathrm{i}\varepsilon^A Q_A + \mathrm{i}\overline{\varepsilon}_{A'}\overline{Q}^{A'}\right) , \tag{3.33}$$

acting on an element of the super-Poincaré group, viz.,

$$g(x^\mu, \theta_A, \overline{\theta}_{A'}) \equiv \exp\left(-\mathrm{i}x^\mu \mathcal{P}_\mu + \mathrm{i}\theta^A Q_A + \mathrm{i}\overline{\theta}_{A'}\overline{Q}^{A'}\right) ,$$

to give

$$\exp\left(-\mathrm{i}a^\mu \mathcal{P}_\mu + \mathrm{i}\varepsilon^A Q_A + \mathrm{i}\overline{\varepsilon}_{A'}\overline{Q}^{A'}\right) \exp\left(-\mathrm{i}x^\mu \mathcal{P}_\mu + \mathrm{i}\theta^A Q_A + \mathrm{i}\overline{\theta}_{A'}\overline{Q}^{A'}\right) . \tag{3.34}$$

We now apply the Baker–Campbell–Hausdorff (BCH) formula

$$\mathrm{e}^A \mathrm{e}^B = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}\big[A, [A, B]\big] - \frac{1}{12}\big[B, [B, A]\big] + \cdots\right) ,$$

to get for (3.34)

$$\begin{aligned} \exp\Big[- \mathrm{i}\left(a^\mu + x^\mu\right)\mathcal{P}_\mu + \mathrm{i}\left(\xi^A + \theta^A\right) Q_A + \mathrm{i}\left(\overline{\varepsilon}_{A'} + \overline{\theta}_{A'}\right)\overline{Q}^{A'} \\ - \left(\varepsilon\sigma^\mu\overline{\theta}\right)\mathcal{P}_\mu + \left(\theta\sigma^\mu\overline{\varepsilon}\right)\mathcal{P}_\mu\Big] . \end{aligned} \tag{3.35}$$

From (3.34) and

$$\mathrm{e}^{-\mathrm{i}a^\mu \mathcal{P}_\mu + \mathrm{i}\varepsilon^A Q_A + \mathrm{i}\overline{\varepsilon}_{A'}\overline{Q}^{A'}} g(x^\mu, \theta_A, \overline{\theta}_{A'}) = g(x'^\mu, \theta'_A, \overline{\theta}_{A'}{}') ,$$

the translations in superspace are extracted as

---

[6] Restricting herewith to $N = 1$ superspace and superfields.

$$x'^{\mu} \equiv x^{\mu} + a^{\mu} - \mathrm{i}\left(\varepsilon\sigma^{\mu}\overline{\theta}\right) + \mathrm{i}\left(\theta\sigma^{\mu}\overline{\varepsilon}\right) , \tag{3.36}$$

$$\theta'_A \equiv \theta_A + \varepsilon_A , \tag{3.37}$$

$$\overline{\theta}'_{A'} \equiv \overline{\theta}_{A'} + \overline{\varepsilon}_{A'} . \tag{3.38}$$

We thus obtain translations in superspace, and with the particularity that translations in the fermionic variables yield a translation component in the spacetime coordinates! It can now be established that the motion in (3.36), (3.37), and (3.38) can be induced by the following differential operators[7]:

$$Q_A = \frac{\partial}{\partial\theta^A} - \mathrm{i}\sigma^{\mu}_{AB'}\overline{\theta}^{B'}\partial_{\mu} , \tag{3.39}$$

$$\overline{Q}^{B'} = \frac{\partial}{\partial\overline{\theta}^{B'}} - \mathrm{i}\theta^C\sigma^{\mu B'}_{C}\partial_{\mu} . \tag{3.40}$$

The subsequent calculation of $\delta\theta_A = (\mathrm{i}\varepsilon^B Q_B)\theta_A$ and its conjugate together with $\delta x^{\mu} = \big(- \mathrm{i}a^{\mu}\mathcal{P}_{\mu} + \mathrm{i}\varepsilon^A Q_A + \mathrm{i}\overline{\varepsilon}_{A'}\overline{Q}^{A'}\big)x^{\mu}$, agree with (3.36), (3.37), and (3.38), whence we do indeed have $\{Q_A, \overline{Q}_{B'}\} = 2\sigma^{\mu}_{AB'}\mathcal{P}_{\mu} \longmapsto 2\mathrm{i}\sigma^{\mu}_{AB'}\partial_{\mu}$. This is the second step in specifying how SUSY operators would act as $Q_A \sim \partial_{\theta_A}$ and $\overline{Q}_{A'} \sim \partial_{\overline{\theta}_{A'}}$.

In summary, in the Poincaré algebra, the four generators $\mathcal{P}_{\mu}$ (admitting a representation as differential operators) correspond to spacetime translations for the coordinates x^{μ}. The structure of the SUSY superspace admits a similar structure for the Grassmanian variables $\theta, \overline{\theta}$ with SUSY generators the Q_A and their Hermitian conjugates $\overline{Q}_{A'} \equiv (Q_A)^{\dagger}$ as differential operators on superspace. By means of $\mathrm{i}\varepsilon^A Q_A$, a translation in θ^A is generated by a constant infinitesimal spinor ε^A plus some transformation[8] in x^{μ}.

Note 3.5 Let us meanwhile point out some technical features and rules characteristic of Grassmanian quatitities, which will be put to use in the following [1, 7, 8, 10]:

- Spinors θ_A and $\overline{\theta}_{A'}$ are mutually anticommuting, i.e. $\theta^0\theta^1 = -\theta^1\theta^0$.
- Consequently, in bilinear terms, the contraction to follow is $\theta\theta = \theta^A\theta_A = -2\theta^0\theta^1 = +2\theta_1\theta_0 = -2\theta_0\theta_1$, and $\overline{\theta\theta} = \overline{\theta}_{A'}\theta^{A'} = 2\overline{\theta}_{0'}\overline{\theta}_{1'}$.

[7] Taking the transformations as $\delta_{\varepsilon}\mathsf{F}(x, \theta, \overline{\theta}) \equiv (\varepsilon Q + \overline{\varepsilon}\overline{Q})\mathsf{F}$ ranging from the transformation $\mathsf{F}(x, \theta, \overline{\theta}) \equiv \exp(\varepsilon Q + \overline{\varepsilon}\overline{Q})\varphi$, where φ is a component multiplet in a set of fields $\{\varphi, \chi, \ldots\}$. See Note 3.6.

[8] A spacetime translation is eventually retrieved through the SUSY algebra. The commutator of two SUSY transformations is a translation in spacetime.

- The reader should note the important and useful property that, since the θ anticommute, any product involving more than two θ or more than two $\overline{\theta}$ vanishes.
- Useful identities are:

$$\theta^A\theta^B = -\frac{1}{2}\varepsilon^{AB}\theta\theta\,, \tag{3.41}$$

$$\overline{\theta}^{A'}\overline{\theta}^{B'} = \frac{1}{2}\varepsilon^{A'B'}\overline{\theta\theta}\,, \tag{3.42}$$

$$\theta_A\theta_B = \frac{1}{2}\varepsilon_{AB}\theta\theta\,, \tag{3.43}$$

$$\overline{\theta}_{A'}\overline{\theta}_{B'} = -\frac{1}{2}\varepsilon_{A'B'}\overline{\theta\theta}\,, \tag{3.44}$$

$$\theta\sigma^\mu\overline{\theta}\,\theta\sigma^\nu\overline{\theta} = -\frac{1}{2}\theta\theta\,\overline{\theta\theta}g^{\mu\nu}\,, \tag{3.45}$$

$$(\theta\psi)\,(\theta\chi) = -\frac{1}{2}(\theta\theta)\,(\psi\chi)\,. \tag{3.46}$$

- Derivatives involving θ and $\overline{\theta}$ are defined by

$$\frac{\partial}{\theta^A}\theta^B \equiv \delta_A{}^B\,, \qquad \frac{\partial}{\overline{\theta}^{A'}}\overline{\theta}^{B'} \equiv \delta_{A'}{}^{B'}\,, \qquad \varepsilon^{AB}\frac{\partial}{\partial^B} = -\frac{\partial}{\partial_A}\,. \tag{3.47}$$

- Integration for a single Grassmannian variable, e.g., θ^1, gives

$$\int \mathrm{d}\theta^1(a+\theta^1 b) = b\,,$$

since $\int \mathrm{d}\theta^0\mathrm{d}\theta^1\theta^1\theta^0 = 1$. Moreover, from $\theta\theta = 2\theta^1\theta^0$, and with the definition $\mathrm{d}^2\theta = \mathrm{d}\theta^1\mathrm{d}\theta^2/2$, it follows that

$$\int \mathrm{d}^2\theta\,\theta\theta = \int \mathrm{d}^2\overline{\theta}\,\overline{\theta\theta} = 1\,, \tag{3.48}$$

with

$$\varepsilon^{AB}\frac{\partial}{\partial\theta^A}\frac{\partial}{\partial\theta^B}\theta\theta = 4\,, \qquad \varepsilon^{A'B'}\frac{\partial}{\partial\overline{\theta}^{A'}}\frac{\partial}{\partial\overline{\theta}^{B'}}\overline{\theta\theta} = 4\,, \tag{3.49}$$

and

$$\mathrm{d}^4\theta \equiv \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta} \longrightarrow \int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\;\theta\theta\overline{\theta\theta} = 1\,. \tag{3.50}$$

Note 3.6 A *superfield* $\mathsf{F}(x, \theta, \overline{\theta})$ is a field defined on superspace, or more precisely, an arbitrary (scalar) function on superspace. It can contain more than one multiplet (possibly within a reducible representation of SUSY), with the irreducible terms being retrieved and related by imposing appropriate contraints. The *chiral* supermultiplet will require a covariant derivative of $\mathsf{F}(x, \theta, \overline{\theta})$ to vanish, while for the *vector* supermultiplet, a reality condition $\mathsf{F}(x, \theta, \overline{\theta}) = F^{\dagger}(x, \theta, \overline{\theta})$ is imposed.

With the SUSY generators Q_A and their Hermitian conjugates $\overline{Q}^{A'} = (Q_A)^{\dagger}$ as differential operators on superspace, using $\mathrm{i}\varepsilon^A Q_A$ to generate a translation in θ^A by a constant infinitesimal spinor ε^A plus some transformation in x^{μ}, we write for Q_A (see [47]),

$$(1 + \varepsilon Q)\mathsf{F}(x, \theta, \overline{\theta}) \equiv \mathsf{F}(x + \delta x, \theta + \varepsilon, \overline{\theta}) , \tag{3.51}$$

or more generally for a SUSY variation of F,

$$\delta_{\varepsilon,\overline{\varepsilon}}\mathsf{F} = (\mathrm{i}\varepsilon Q + \mathrm{i}\overline{\varepsilon}\overline{Q})\mathsf{F} , \tag{3.52}$$

to retrieve the action on F as

$$\left(1+\mathrm{i}\varepsilon Q+\mathrm{i}\overline{\varepsilon}\overline{Q}\right)\mathsf{F}(x, \theta, \overline{\theta}) = \mathsf{F}\left(x^{\mu} - \mathrm{i}\varepsilon\sigma^{\mu}\overline{\theta} + \mathrm{i}\theta\sigma^{\mu}\overline{\varepsilon}, \theta^A + \varepsilon^A, \overline{\theta}^{A'} + \overline{\varepsilon}^{A'}\right) . \tag{3.53}$$

In addition, for a Taylor expansion (terminating at order $\theta\theta\overline{\theta}\overline{\theta}$), we can write generally

$$\begin{aligned} \mathsf{F}^{\mathsf{I}}(x, \theta, \overline{\theta}) = {} & \phi^{\mathsf{I}}(x) + \theta\psi^{\mathsf{I}}(x) + \overline{\theta}\overline{\chi}^{\mathsf{I}}(x) + \theta\theta m^{\mathsf{I}}(x) + \overline{\theta}\overline{\theta}n^{\mathsf{I}}(x) \\ & + \theta\sigma^{\mu}\overline{\theta}v^{\mathsf{I}}_{\mu} + \theta\theta\overline{\theta}\overline{\lambda}^{\mathsf{I}}(x) + \overline{\theta}\overline{\theta}\theta\rho^{\mathsf{I}}(x) + \theta\theta\overline{\theta}\overline{\theta}d^{\mathsf{I}}(x) , \end{aligned} \tag{3.54}$$

recalling that I is an arbitrary group index. This suggests a multiplet containing *four* scalars ($\phi^{\mathsf{I}}, m^{\mathsf{I}}, n^{\mathsf{I}}, d^{\mathsf{I}}$), *four* spinors ($\psi^{\mathsf{I}}, \overline{\chi}^{\mathsf{I}}, \overline{\lambda}^{\mathsf{I}}, \rho^{\mathsf{I}}$) and *one* vector ($v^{\mathsf{I}}_{\mu}$). Now from a superfield $\mathsf{F}^I(x, \theta, \overline{\theta})$, this means far more degrees of freedom than present for example in the Wess–Zumino model, where we use a chiral multiplet. As already indicated, we therefore need a procedure to produce a chiral multiplet from superfields within superspace. A suitable tool follows.

Note 3.7 From the operators (3.39) and (3.40), SUSY *covariant* derivatives D_A and $\overline{\mathsf{D}}_{B'}$ can be defined (noting that, for a superfield F, $\partial_A\mathsf{F}$ is *not* a superfield), these anticommuting with the SUSY operators Q and $\overline{Q}$, so that $\mathsf{D}_A\mathsf{F} = 0$ or $\overline{\mathsf{D}}_{B'}\mathsf{F} = 0$ would constitute SUSY invariant constraints. (With $\delta_{\varepsilon,\overline{\varepsilon}}(\mathsf{D}_A\mathsf{F}) = \mathsf{D}_A(\delta_{\varepsilon,\overline{\varepsilon}}\mathsf{F})$ and $\delta_{\varepsilon,\overline{\varepsilon}}(\overline{\mathsf{D}}_{A'}\mathsf{F}) = \overline{\mathsf{D}}_{A'}(\delta_{\varepsilon,\overline{\varepsilon}}\mathsf{F})$.) One thus has a SUSY covariant procedure to reduce the number of components in a particular superfield, and thereby define a specific theory. Hence, we find [1, 7, 11]

$$\mathsf{D}_A \equiv \frac{\partial}{\partial \theta^A} + \mathrm{i}\sigma^{\mu}_{AB'}\overline{\theta}^{B'}\partial_\mu\,, \tag{3.55}$$

$$\overline{\mathsf{D}}_{B'} \equiv (\mathsf{D}_B)^\dagger \equiv -\frac{\partial}{\partial \overline{\theta}^{B'}} - \mathrm{i}\theta^B\sigma^{\mu}_{BB'}\partial_\mu\,, \tag{3.56}$$

with

$$\left\{\mathsf{D}_A, \overline{\mathsf{D}}_{B'}\right\} = -2\mathrm{i}\sigma^{\mu}_{AB'}\partial_\mu\,, \tag{3.57}$$

$$\left\{\mathsf{D}_A, \mathsf{D}_B\right\} = \left\{\overline{\mathsf{D}}_{A'}, \overline{\mathsf{D}}_{B'}\right\} = 0\,, \tag{3.58}$$

$$\left\{\mathsf{D}_A, Q_B\right\} = \left\{\overline{\mathsf{D}}_{B'}, Q_A\right\} = \left\{\mathsf{D}_A, \overline{Q}_{B'}\right\} = \left\{\overline{\mathsf{D}}_{B'}, \overline{Q}_{A'}\right\} = 0\,. \tag{3.59}$$

3.3.1 Chiral Superfields

A commonly used application of the above covariant derivatives is indeed to define a chiral or an anti-chiral superfield, Φ or $\overline{\Phi}$, respectively, by (see, e.g., [47])

$$\overline{\mathsf{D}}_{B'}\Phi = 0\,, \tag{3.60}$$

$$\mathsf{D}_A\overline{\Phi} = 0\,, \tag{3.61}$$

leading, in component expansion, to[9]

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta f(y)\,, \tag{3.62}$$

$$\overline{\Phi}(y,\overline{\theta}) = \overline{\phi}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) + \overline{\theta}\,\overline{\theta}\,\overline{f}(\overline{y})\,, \tag{3.63}$$

or with a further Taylor expansion now in terms of x, θ, and $\overline{\theta}$:

$$\begin{aligned}\Phi(y,\theta) \simeq{}& \phi(x) + \sqrt{2}\theta\psi(x) + \mathrm{i}\theta\sigma^{\mu}\overline{\theta}\partial_\mu\phi(x) + \theta\theta f(x) \\ &- \frac{\mathrm{i}}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^{\mu}\overline{\theta} + \frac{1}{4}\theta\theta\overline{\theta}\,\overline{\theta}\partial^2\phi(x)\,,\end{aligned} \tag{3.64}$$

[9] Φ depends only on θ and y^μ, i.e., all $\overline{\theta}$ dependence is through y^μ, and $\overline{\Phi}$ depends only on $\overline{\theta}$ and $\overline{y}^\mu$.

$$\overline{\Phi}(y,\overline{\theta}) \simeq \overline{\phi}(x) + \sqrt{2}\overline{\theta}\,\overline{\psi}(x) - \mathrm{i}\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{\phi}(x) + \overline{\theta}\,\overline{\theta}\,\overline{f}(x) \\ + \frac{\mathrm{i}}{\sqrt{2}}\overline{\theta}\,\overline{\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}(x) + \frac{1}{4}\theta\theta\overline{\theta}\,\overline{\theta}\partial^2\overline{\phi}(x)\,, \tag{3.65}$$

due to

$$\mathsf{D}_A\overline{\theta} = \overline{\mathsf{D}}_{B'}\theta = \mathsf{D}_A\overline{y}^\mu = \overline{\mathsf{D}}_{B'}y^\mu = 0\,, \tag{3.66}$$

$$y^\mu \equiv x^\mu + \mathrm{i}\theta\sigma^\mu\overline{\theta}\,, \tag{3.67}$$

$$\overline{y}^\mu \equiv x^\mu - \mathrm{i}\theta\sigma^\mu\overline{\theta}\,. \tag{3.68}$$

From (3.64), a chiral superfield describes one complex scalar ϕ and one Weyl fermion ψ. The field f constitutes an *auxiliary* field (see Sect. 3.2.3). But how specifically do the fields constituting Φ transform under SUSY? For chiral superfields, changing to variables $y^\mu, \theta, \overline{\theta}$, where

$$Q_A = \frac{\partial}{\partial\theta^A}\,, \qquad \overline{Q}_{B'} = -\frac{\partial}{\partial\overline{\theta}^{B'}} + 2\mathrm{i}\theta^B\sigma^\mu_{BB'}\frac{\partial}{\partial y^\mu}\,, \tag{3.69}$$

this induces

$$\delta\Phi(y,\theta) \equiv \left(\varepsilon Q + \overline{\varepsilon}\overline{Q}\right)\Phi(y,\theta) = \left(\varepsilon^A\frac{\partial}{\partial\theta^A} + 2\mathrm{i}\theta\sigma^\mu\overline{\varepsilon}\frac{\partial}{\partial y^\mu}\right)\Phi(y,\theta) \\ = \sqrt{2}\varepsilon\psi + \sqrt{2}\theta\left(\sqrt{2}\varepsilon f + \sqrt{2}\mathrm{i}\sigma^\mu\overline{\varepsilon}\partial_\mu\phi\right) + \theta\theta\left(-\mathrm{i}\sqrt{2}\overline{\varepsilon}\overline{\sigma}^\mu\partial_\mu\psi\right)\,, \tag{3.70}$$

from which the SUSY transformations of the component fields follow, up to a rescaling by constant factors (see Sect. 3.2.3):

$$\delta\phi = \sqrt{2}\varepsilon\psi\,, \tag{3.71}$$

$$\delta f = \sqrt{2}\mathrm{i}\overline{\varepsilon}\partial_\mu\psi\sigma^\mu\,, \tag{3.72}$$

$$\delta\psi = \sqrt{2}\mathrm{i}\partial_\mu\phi\sigma^\mu\overline{\varepsilon} + \sqrt{2}f\varepsilon\,. \tag{3.73}$$

The (SUSY) superspace framework is therefore expected to be quite efficient and economical in producing SUSY invariant actions. The reader may verify this through Exercise 3.6, by analysing how SUSY invariant actions can be built from (SUSY) superspace and superfields, therefore retrieving the Wess–Zumino model as a particular case.

Note 3.8 We can go beyond the framework of the Wess–Zumino model introduced above, and not just in terms of the choice of superpotential. Within a renormalizable theory and for chiral superfields Φ, this means cubic superpotentials, leading to quartic scalar potentials, and some constant K for the *metric* of the kinetic part. A less obvious step, but one that we will explore in SQC, is, more precisely, to replace $\Phi_{\mathsf{I}}^{\dagger}\Phi_{\mathsf{I}}$ by kinetic terms of the form $\mathsf{K}^{\mathsf{I}}_{\mathsf{J}}\Phi_{\mathrm{I}}^{\dagger}\Phi^{\mathsf{J}}$, where K is a Hermitian matrix. (If K is constant, after diagonalising and rescaling the fields, this then reduces to the canonical kinetic term $\Phi_{\mathsf{I}}^{\dagger}\Phi^{\mathsf{I}}$.) K is retrieved from a real Kähler potential $\mathsf{K}(\Phi_{\mathsf{I}}^{\dagger}, \Phi_{\mathsf{J}})$ (see Note 3.9).

Phenomenologically, we want an *effective* theory, valid at low energies only, with *whatever it may contain* at higher energies, as long as it approaches (within falsifiable deviations) a phenomenological low energy limit. Hence, the study of the *supersymmetric non-linear sigma model* [2, 3, 12].

Proceeding to a wider context than in (3.32), with an action

$$S = \int d^4x \left[\int d^2\theta d^2\overline{\theta}\, \mathsf{K}(\Phi^{\mathsf{I}}, \Phi_{\mathsf{I}}^{\dagger}) + \int d^2\theta \mathsf{W}(\Phi^{\mathsf{I}}) + \int d^2\overline{\theta}\, \mathsf{W}^{\dagger}(\Phi_{\mathsf{I}}^{\dagger}) \right], \tag{3.74}$$

from which a larger domain is retrieved for exploration (and this is justifiable, because some such actions arise from SUGRA). In a few basic steps, a recipe for this would be:

- Expand

$$\mathsf{W}(\Phi) \equiv \mathsf{W}(\phi) + \mathsf{W}_{\mathsf{I}}\Delta^{\mathsf{I}} + \frac{1}{2}\mathsf{W}_{\mathsf{IJ}}\Delta^{\mathsf{I}}\Delta^{\mathsf{J}}, \tag{3.75}$$

$$\Delta^{\mathsf{I}}(y) \equiv \Phi^{\mathsf{I}} - \phi^{\mathsf{I}}(y) = \sqrt{2}\theta\psi^{\mathsf{I}}(y) + \theta\theta f^{\mathsf{I}}(y), \tag{3.76}$$

  whose $\theta\theta$-components give

$$\int d^2\theta \mathsf{W}(\Phi^{\mathrm{I}}) + \text{h.c.} = \left(-\mathsf{W}_{\mathsf{I}} f^{\mathsf{I}} - \frac{1}{2}\mathsf{W}_{\mathsf{IJ}}\psi^{\mathsf{I}}\psi^{\mathsf{J}} \right) + \text{h.c.} \tag{3.77}$$

- Introduce a function $\mathsf{K}(\Phi^{\mathsf{I}}, \Phi_{\mathsf{I}}^{\dagger})$ as a real superfield, so that $\overline{\mathsf{K}}(\phi^{\mathsf{I}}, \phi_{\mathsf{J}}^{\dagger}) = \mathsf{K}(\phi_{\mathsf{I}}^{\dagger}, \phi^{\mathsf{J}})$. In addition,

$$\mathsf{K}_I \equiv \frac{\partial}{\partial\phi^{\mathsf{I}}}\mathsf{K}(\phi, \phi^{\dagger}), \quad \mathsf{K}^{\mathsf{J}} \equiv \frac{\partial}{\partial\phi_{\mathsf{J}}^{\dagger}}\mathsf{K}(\phi, \phi^{\dagger}), \quad \mathsf{K}_{\mathsf{I}}^{\mathsf{J}} \equiv \frac{\partial^2}{\partial\phi^{\mathsf{I}}\partial\phi_{\mathsf{J}}^{\dagger}}\mathsf{K}(\phi, \phi^{\dagger}). \tag{3.78}$$

- Perform a Taylor expansion of $\mathsf{K}(\Phi^{\mathrm{I}}, \Phi_{\mathrm{I}}^{\dagger})$:

$$\begin{aligned}\mathsf{K}(\Phi^{\mathsf{I}}, \Phi_{\mathsf{I}}^{\dagger}) = \mathsf{K}(\phi^{\mathsf{I}}, \phi_{\mathsf{I}}^{\dagger}) + \mathsf{K}_{\mathsf{I}}\Delta^{\mathsf{I}} + \mathsf{K}^{\mathsf{I}}\Delta_{\mathsf{I}}^{\dagger} \\ +\frac{1}{2}\mathsf{K}_{\mathsf{IJ}}\Delta^{\mathsf{I}}\Delta^{\mathsf{J}} + \frac{1}{2}\mathsf{K}^{\mathsf{IJ}}\Delta_{\mathsf{I}}^{\dagger}\Delta_{\mathsf{J}}^{\dagger} + \mathsf{K}_{\mathsf{I}}^{\mathsf{J}}\Delta^{\mathsf{I}}\Delta_{\mathsf{J}}^{\dagger} \\ +\frac{1}{2}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{K}}\Delta^{\mathsf{I}}\Delta^{\mathsf{J}}\Delta_{\mathsf{K}}^{\dagger} + \frac{1}{2}\mathsf{K}_{\mathsf{K}}^{\mathsf{IJ}}\Delta_{\mathsf{I}}^{\dagger}\Delta_{\mathsf{J}}^{\dagger}\Delta^{\mathsf{K}} + \frac{1}{4}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{KL}}\Delta^{\mathsf{I}}\Delta^{\mathsf{J}}\Delta_{\mathsf{K}}^{\dagger}\Delta_{\mathsf{L}}^{\dagger} .\end{aligned} \tag{3.79}$$

- For the coefficient of $\theta\theta\overline{\theta}\,\overline{\theta}$, we obtain:

$$\begin{aligned}&\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\, \mathsf{K}(\Phi^{\mathsf{I}}, \Phi_{\mathsf{I}}^{\dagger}) \\ &= -\frac{1}{4}\mathsf{K}_{\mathsf{I}}\partial^2\phi^{\mathsf{I}} - \frac{1}{4}\mathsf{K}^{\mathsf{I}}\partial^2\phi_{\mathsf{I}}^{\dagger} - \frac{1}{4}\mathsf{K}_{\mathsf{IJ}}\partial_\mu\phi^{\mathsf{I}}\partial^\mu\phi^{\mathsf{J}} + \text{h.c.} \\ &+ \mathsf{K}_{\mathsf{I}}^{\mathsf{J}}\left(f^{\mathsf{I}}f_{\mathsf{J}}^{\dagger} + \frac{1}{2}\partial_\mu\phi^{\mathsf{I}}\partial^\mu\phi_{\mathsf{J}}^{\dagger} - \frac{\mathrm{i}}{2}\psi^{\mathsf{I}}\sigma^\mu\partial_\mu\overline{\psi}_{\mathsf{J}} + \frac{\mathrm{i}}{2}\partial_\mu\psi^{\mathsf{I}}\sigma^\mu\overline{\psi}_{\mathsf{J}}\right) \\ &+ \frac{\mathrm{i}}{4}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{K}}\left(\psi^{\mathsf{I}}\sigma^\mu\overline{\psi}_{\mathsf{K}}\partial_\mu\phi^{\mathsf{J}} + \psi^{\mathsf{J}}\sigma^\mu\overline{\psi}_{\mathsf{K}}\partial_\mu\phi^{\mathsf{I}} - 2\mathrm{i}\psi^{\mathsf{I}}\psi^{\mathsf{J}}f_{\mathsf{K}}^{\dagger}\right) + \text{h.c.} \\ &+ \frac{1}{4}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{KL}}\psi^{\mathsf{I}}\psi^{\mathsf{J}}\,\overline{\psi}_{\mathsf{K}}\overline{\psi}_{\mathsf{L}} ,\end{aligned} \tag{3.80}$$

with

$$\begin{aligned}\partial_\mu\partial^\mu\mathsf{K}(\phi^{\mathsf{I}}, \phi_{\mathsf{J}}^{\dagger}) = \mathsf{K}_{\mathsf{I}}\partial^2\phi^{\mathsf{I}} + \mathsf{K}^{\mathsf{I}}\partial^2\phi_{\mathsf{I}}^{\dagger} + 2\mathsf{K}_{\mathsf{I}}^{\mathsf{J}}\partial_\mu\phi_{\mathsf{J}}^{\dagger}\partial^\mu\phi^{\mathsf{I}} \\ +\mathsf{K}_{\mathsf{IJ}}\partial_\mu\phi^{\mathsf{I}}\partial^\mu\phi^{\mathsf{J}} + \mathsf{K}^{\mathsf{IJ}}\partial_\mu\phi_{\mathsf{I}}^{\dagger}\partial^\mu\phi_{\mathsf{J}}^{\dagger} .\end{aligned} \tag{3.81}$$

- In the end, it becomes

$$\begin{aligned}&\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\mathsf{K}(\Phi^{\mathsf{I}}, \Phi_{\mathsf{I}}^{\dagger}) \\ &= \mathsf{K}_{\mathsf{I}}^{\mathsf{J}}\left(f^{\mathsf{I}}f_{\mathsf{J}}^{\dagger} - \partial_\mu\phi^{\mathsf{I}}\partial^\mu\phi_{\mathsf{J}}^{\dagger} + \frac{\mathrm{i}}{2}\psi^{\mathsf{I}}\overline{\sigma}^\mu\partial_\mu\overline{\psi}_{\mathsf{J}} + \frac{\mathrm{i}}{2}\partial_\mu\psi^{\mathsf{I}}\overline{\sigma}^\mu\overline{\psi}_{\mathsf{J}}\right) \\ &+ \frac{\mathrm{i}}{4}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{K}}\left(\psi^{\mathsf{I}}\sigma^\mu\overline{\psi}_{\mathsf{K}}\partial_\mu\phi^{\mathsf{J}} + \psi^{\mathsf{J}}\sigma^\mu\overline{\psi}_{\mathsf{K}}\partial_\mu\phi^{\mathsf{I}} - 2\mathrm{i}\psi^{\mathsf{I}}\psi^{\mathsf{J}}f_{\mathsf{K}}^{\dagger}\right) + \text{h.c.} \\ &+ \frac{1}{4}\mathsf{K}_{\mathsf{IJ}}^{\mathsf{KL}}\psi^{\mathsf{I}}\psi^{\mathsf{J}}\,\overline{\psi}_{\mathsf{K}}\overline{\psi}_{\mathsf{L}} - \frac{1}{4}\partial_\mu\partial^\mu\mathsf{K}(\phi^{\mathsf{I}}, \phi_{\mathsf{J}}^{\dagger}) .\end{aligned} \tag{3.82}$$

Note 3.9 The Kähler framework is much more than a mere extension of the components of the SUSY Lagrangian. For example, consider the transformation

$$\mathsf{K}(\phi, \phi^\dagger) \longrightarrow \mathsf{K}(\phi, \phi^\dagger) + \mathrm{g}(\phi) + \overline{\mathrm{g}}(\phi^\dagger) \,. \tag{3.83}$$

Concerning (3.82), as long as the last term (a total derivative) is absent and the mixed terms with at least one upper and one lower index are present, (3.83) will not affect the Lagrangian. In fact, it allows one to introduce a metric from within the kinetic terms for the complex scalars, viz.,

$$\mathsf{K}_\mathsf{I}^\mathsf{J} \equiv \frac{\partial^2}{\partial \phi^\mathsf{I} \partial \phi_\mathsf{J}^\dagger} \mathsf{K}(\phi, \phi^\dagger) \,, \tag{3.84}$$

called a Kähler metric. Accordingly, the scalar function $\mathsf{K}(\phi, \phi^\dagger)$ constitutes the Kähler potential. The Kähler metric is of course invariant under Kähler transformations (3.83), establishing the complex scalars ϕ^I as (local) complex coordinates on a Kähler manifold (what else?). In more mathematical terms, the *target* manifold of the application is a *sigma-model*, with Kähler properties (see Note 3.8).

Note that the Kähler invariance (3.83) at superfield level becomes

$$\mathsf{K}(\Phi, \Phi^\dagger) \to \mathsf{K}(\Phi, \Phi^\dagger) + \mathrm{g}(\Phi) + \overline{\mathrm{g}}(\Phi^\dagger) \,, \tag{3.85}$$

where $\mathrm{g}(\Phi)$ is a chiral superfield and its $\theta\theta\overline{\theta}\,\overline{\theta}$ component is a total derivative, whence $\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} \mathrm{g}(\Phi) = \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\, \overline{\mathrm{g}}(\Phi^\dagger) = 0$.

With $\mathsf{K}_\mathsf{I}^\mathsf{J}$ as a metric, the corresponding affine connection and curvature tensors follow. With upper and lower indices to denote derivatives with respect to ϕ^I or $\phi_\mathsf{J}^\dagger$, an often used convention for the metric is, with $\phi_\mathsf{J}^\dagger \to \phi^{\overline{\mathsf{J}}}$, $\mathsf{K}_\mathsf{I}^\mathsf{J} \to \mathsf{K}_{\mathsf{I}\overline{\mathsf{J}}}$, and then

$$\begin{aligned} \mathsf{K}_{\mathsf{I}\overline{\mathsf{J}}} &= \mathsf{K}_{\overline{\mathsf{J}}\mathsf{I}} \,, \qquad \mathsf{K}_{\mathsf{I}\mathsf{J}} = \mathsf{K}_{\overline{\mathsf{I}\mathsf{J}}} = 0 \,, \\ \mathsf{K}^{\mathsf{I}\overline{\mathsf{J}}} &= \mathsf{K}^{\overline{\mathsf{J}}\mathsf{I}} \,, \qquad \mathsf{K}^{\mathsf{I}\mathsf{J}} = \mathsf{K}^{\overline{\mathsf{I}\mathsf{J}}} = 0 \,. \end{aligned} \tag{3.86}$$

The affine connection is given by

$$\Gamma^\mathsf{L}_{\mathsf{I}\mathsf{J}} = \mathsf{K}^{\mathsf{L}\overline{\mathsf{M}}} \mathsf{K}_{\mathsf{I}\mathsf{J}\overline{\mathsf{M}}} \,, \qquad \Gamma^{\overline{\mathsf{L}}}_{\overline{\mathsf{I}\mathsf{J}}} = \mathsf{K}^{\overline{\mathsf{L}\mathsf{M}}} \mathsf{K}_{\overline{\mathsf{I}\mathsf{J}\mathsf{M}}} \,, \tag{3.87}$$

and all other components with mixed indices vanish. The only nonvanishing components in the Kähler curvature tensor are

$$(\mathsf{R}_{\overline{\mathsf{K}}\mathsf{I}})^\mathsf{L}{}_\mathsf{J} = \partial_{\overline{K}} \Gamma^\mathsf{L}_{\mathsf{I}\mathsf{J}} = \mathsf{K}^{\mathsf{L}\overline{\mathsf{P}}} \left(\mathsf{K}_{\mathsf{I}\mathsf{J}\overline{\mathsf{P}\mathsf{K}}} - \mathsf{K}_{\mathsf{I}\mathsf{J}\overline{\mathsf{M}}} \mathsf{K}^{\overline{\mathsf{M}}\mathsf{N}} \mathsf{K}_{\mathsf{N}\overline{\mathsf{P}\mathsf{K}}} \right) \,, \tag{3.88}$$

with of course $(\mathsf{R}_{\mathsf{I}\overline{\mathsf{K}}})^{\mathsf{L}}{}_{\mathsf{J}} = -(\mathsf{R}_{\overline{\mathsf{K}}\mathsf{I}})^{\mathsf{L}}{}_{\mathsf{J}}$, and similarly

$$(\mathsf{R}_{\mathsf{I}\overline{\mathsf{K}}})^{\overline{\mathsf{L}}}_{\overline{\mathsf{J}}} = -(\mathsf{R}_{\overline{\mathsf{K}}\mathsf{I}})^{\overline{\mathsf{L}}}_{\overline{\mathsf{J}}} = \partial_{\mathsf{I}}\Gamma^{\overline{\mathsf{L}}}_{\overline{\mathsf{K}\mathsf{J}}} = \mathsf{K}^{\overline{\mathsf{L}}\mathsf{P}}\left(\mathsf{K}_{\mathsf{I}\mathsf{P}\overline{\mathsf{K}\mathsf{J}}} - \mathsf{K}_{\mathsf{I}\mathsf{P}\overline{\mathsf{M}}}\mathsf{K}^{\overline{\mathsf{M}}\mathsf{N}}\mathsf{K}_{\mathsf{N}\overline{\mathsf{K}\mathsf{J}}}\right) . \tag{3.89}$$

In the previous more geometric (and perhaps simpler) notation,

$$\mathsf{K}_{\mathsf{I}\overline{\mathsf{J}}} \to \mathsf{K}^{\mathsf{J}}_{\mathsf{I}}, \qquad \Gamma^{\mathsf{L}}_{\mathsf{I}\mathsf{J}} \to \Gamma^{\mathsf{L}}_{\mathsf{I}\mathsf{J}}, \qquad \Gamma^{\overline{\mathsf{L}}}_{\overline{\mathsf{I}\mathsf{J}}} \to \Gamma^{\mathsf{I}\mathsf{J}}_{\mathsf{L}}, \qquad (\mathsf{R}_{\overline{\mathsf{K}}\mathsf{I}})_{\overline{\mathsf{L}}\mathsf{J}} \to \mathsf{R}^{\mathsf{K}\mathsf{L}}_{\mathsf{I}\mathsf{J}}, \tag{3.90}$$

$$\Gamma^{\mathsf{L}}_{\mathsf{I}\mathsf{J}} = (\mathsf{K}^{-1})^{\mathsf{L}}_{\mathsf{K}}\mathsf{K}^{\mathsf{K}}_{\mathsf{I}\mathsf{J}}, \qquad \Gamma^{\mathsf{I}\mathsf{J}}_{\mathsf{L}} = (\mathsf{K}^{-1})^{\mathsf{K}}_{\mathsf{L}}\mathsf{K}^{\mathsf{I}\mathsf{J}}_{\mathsf{K}}, \qquad \mathsf{R}^{\mathsf{K}\mathsf{L}}_{\mathsf{I}\mathsf{J}} = \mathsf{K}^{\mathsf{K}\mathsf{L}}_{\mathsf{I}\mathsf{J}} - \mathsf{K}^{\mathsf{M}}_{\mathsf{I}\mathsf{J}}(\mathsf{K}^{-1})^{\mathsf{N}}_{\mathsf{M}}\mathsf{K}^{\mathsf{K}\mathsf{L}}_{\mathsf{N}} . \tag{3.91}$$

As expected, within the Kähler manifold structure, we have *Kähler covariant derivatives* of the fermions:

$$\mathsf{D}_\mu\psi^{\mathsf{I}} = \partial_\mu\psi^{\mathsf{I}} + \Gamma^{\mathsf{I}}_{\mathsf{J}\mathsf{K}}\partial_\mu\phi^{\mathsf{J}}\,\psi^{\mathsf{K}} = \partial_\mu\psi^{\mathsf{I}} + (\mathsf{K}^{-1})^{\mathsf{I}}_{\mathsf{L}}\,K^{\mathsf{L}}_{\mathsf{J}\mathsf{K}}\partial_\mu\phi^{\mathsf{J}}\,\psi^{\mathsf{K}}, \tag{3.92}$$

$$\mathsf{D}_\mu\overline{\psi}_{\mathsf{J}} = \partial_\mu\overline{\psi}_{\mathsf{J}} + \Gamma^{\mathsf{K}\mathsf{I}}_{\mathsf{J}}\partial_\mu\phi^{\dagger}_{\mathsf{K}}\,\overline{\psi}_{\mathsf{I}} = \partial_\mu\overline{\psi}_{\mathsf{J}} + (\mathsf{K}^{-1})^{\mathsf{L}}_{\mathsf{J}}\,\mathsf{K}^{\mathsf{K}\mathsf{I}}_{\mathsf{L}}\partial_\mu\phi^{\dagger}_{\mathsf{K}}\,\overline{\psi}_{\mathsf{I}} . \tag{3.93}$$

Furthermore, when we eliminate the auxiliary fields f^{I} through the equation of motion

$$f^{\mathsf{I}} = -(\mathsf{K}^{-1})^{\mathsf{I}}_{\mathsf{J}}\mathsf{W}^{\mathsf{J}} + \frac{1}{2}\Gamma^{\mathsf{I}}_{\mathsf{J}\mathsf{K}}\psi^{\mathsf{J}}\psi^{\mathsf{K}},$$

the full curvature tensor appears! The rather geometrical and attractive Lagrangian is then [47]

$$\int \mathrm{d}^4x \left\{\int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\mathsf{K}(\Phi,\Phi^\dagger) + \int \mathrm{d}^2\theta\mathsf{W}(\Phi) + \int \mathrm{d}^2\overline{\theta}[\mathsf{W}(\Phi)]^\dagger\right\} \tag{3.94}$$
$$= \int \mathrm{d}^4x\Big[\mathsf{K}^{\mathsf{J}}_{\mathsf{I}}\left(-\partial_\mu\phi^{\mathsf{I}}\partial^\mu\phi^\dagger_{\mathsf{J}} - \mathrm{i}\overline{\psi}^{\mathsf{J}}\overline{\sigma}^\mu\mathsf{D}_\mu\psi^{\mathsf{I}}\right) - (\mathsf{K}^{-1})^{\mathsf{I}}_{\mathsf{J}}\mathsf{W}_{\mathsf{I}}\mathsf{W}^{\mathsf{J}}$$
$$-\frac{1}{2}\left(\mathsf{W}_{\mathsf{I}\mathsf{J}} - \Gamma^{\mathsf{K}}_{\mathsf{I}\mathsf{J}}\mathsf{W}_{\mathsf{K}}\right)\psi^{\mathsf{I}}\psi^{\mathsf{J}} - \frac{1}{2}\left(\mathsf{W}^{\mathsf{I}\mathsf{J}} - \Gamma^{\mathsf{I}\mathsf{J}}_{\mathsf{K}}\mathsf{W}^{\mathsf{K}}\right)\overline{\psi}_{\mathsf{I}}\overline{\psi}_{\mathsf{J}} + \frac{1}{4}\mathsf{R}^{\mathsf{K}\mathsf{L}}_{\mathsf{I}\mathsf{J}}\psi^{\mathsf{I}}\psi^{\mathsf{J}}\overline{\psi}_{\mathsf{K}}\overline{\psi}_{\mathsf{L}}\Big] .$$

3.3.2 Vector Superfields

To make them more realistic, SUSY theories must incorporate supermultiplets of higher spin, e.g., an $N = 1$ vector multiplet. To this end, we employ the generic superfield (3.54) and construct, from $\mathsf{V}(x,\theta,\overline{\theta}) = \mathsf{V}^\dagger(x,\theta,\overline{\theta})$, a *real* superfield with the x^μ expansion

$$\begin{aligned} \mathsf{V}(x,\theta,\bar\theta) \qquad\qquad (3.95)\\ &= C + \mathrm{i}\theta\chi - \mathrm{i}\bar\theta\bar\chi - \theta\sigma^\mu\bar\theta v_\mu + \frac{\mathrm{i}}{2}\theta\theta(M+\mathrm{i}N) - \frac{\mathrm{i}}{2}\bar\theta\bar\theta(M-\mathrm{i}N)\\ &\quad + \mathrm{i}\theta\theta\,\bar\theta\left(\bar\lambda + \frac{\mathrm{i}}{2}\bar\sigma^\mu\partial_\mu\chi\right) - \mathrm{i}\bar\theta\bar\theta\,\theta\left(\lambda + \frac{\mathrm{i}}{2}\sigma^\mu\partial_\mu\bar\chi\right) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(\mathsf{d} + \frac{1}{2}\partial^2 C\right), \end{aligned}$$

with 8 bosonic components (C, d, M, N, v_m) and 8 fermionic components (χ, λ). The reason for the form of (3.95) is that $f_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu$, d, $\bar\lambda$, λ form an irreducible representation of SUSY, with the variation of the d field leading to a total divergence.

Note 3.10 An obvious step is to construct SUSY extensions of *Abelian* gauge invariance and corresponding transformations. And indeed, the transformation

$$\mathsf{V} \to \mathsf{V} + \Phi + \Phi^\dagger\,, \qquad (3.96)$$

with a chiral superfield Φ, accordingly establishes

$$v_\mu \to v_\mu + \partial_\mu(2\mathrm{Im}\phi)\,. \qquad (3.97)$$

This is indeed an Abelian gauge transformation [1]. By including (3.96) as a symmetry, an appropriate (gauge) choice of ϕ would eliminate χ, C, M, N and one component of v_m. This choice is called the *Wess–Zumino gauge*, thus determining

$$\mathsf{V}_{\mathrm{WZ}} \equiv -\theta\sigma^\mu\bar\theta v_\mu(x) + \mathrm{i}\theta\theta\,\bar\theta\,\bar\lambda(x) - \mathrm{i}\bar\theta\,\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\,\bar\theta\mathsf{d}(x)\,, \qquad (3.98)$$

with the rather useful property that only

$$\mathsf{V}_{\mathrm{WZ}}^2 = \theta\sigma^\mu\bar\theta\;\theta\sigma^\nu\bar\theta\; v_\mu v_\nu = -\frac{1}{2}\theta\theta\bar\theta\,\bar\theta\; v_\mu v^\mu \qquad (3.99)$$

is non-vanishing. (In fact, $\mathsf{V}_{\mathrm{WZ}}^n = 0$, $n \geq 3$.)

A consistent field strength (using the Wess–Zumino gauge) follows:

$$\begin{aligned} \mathsf{F}_A^{\mathrm{WZ}} &\sim \bar{\mathsf{D}}\bar{\mathsf{D}}\mathsf{D}_A\mathsf{V} \qquad (3.100)\\ &= -\mathrm{i}\lambda_A(y) + \theta_A\mathsf{d}(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_A f_{\mu\nu}(y) + \theta\theta\left[\sigma^\mu\partial_\mu\bar\lambda(y)\right]_A\,, \end{aligned}$$

$$f_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu\,, \qquad (3.101)$$

where $f_{\mu\nu}$ is the Abelian field strength for v_μ. A gauge invariant SUSY Abelian gauge theory follows. The significant feature is that, as F_A is a chiral superfield, $\int d^2\theta \mathsf{F}^A \mathsf{F}_A$ will be a SUSY invariant Lagrangian, whose $\theta\theta$ term (F-term) is

$$\mathsf{F}^A\mathsf{F}_A\Big|_{\theta\theta} = -2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \mathsf{d}^2 - \frac{1}{2}(\sigma^{\mu\nu})^{AB}(\sigma^{\rho\sigma})_{AB}f_{\mu\nu}f_{\rho\sigma} , \tag{3.102}$$

from which

$$\int d^2\theta \mathsf{F}^A\mathsf{F}_A = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu} - 2\mathrm{i}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \mathsf{d}^2 + \frac{\mathrm{i}}{4}\varepsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma} , \tag{3.103}$$

noting that the d field is auxiliary.

Note 3.11 A generalization of F_A is required to deal with the *non*-Abelian case, where the vector multiplet will include the gauge boson $v_\mu^{(a)}$ and the gaugino $\lambda^{(a)}$:

- First, as expected, the vector multiplet has $\mathsf{V} \equiv \mathsf{V}_{(a)}\mathcal{T}^{(a)}$, $a = 1, \dots \dim G$, where the $\mathcal{T}_{(a)}$ are the generators of the gauge group G in the adjoint representation. The gauge group generators $\mathcal{T}^{(a)}$ satisfy $[\mathcal{T}^{(a)}, \mathcal{T}^{(b)}] = \mathrm{i}\check{f}^{(a)(b)(c)}\mathcal{T}^{(c)}$ with real structure constants $\check{f}^{(a)(b)(c)}$.
- Second, use e^{V} instead of V, with the generalisation of the transformation (3.96):

$$\mathrm{e}^{\mathsf{V}} \longrightarrow \mathrm{e}^{-\mathrm{i}\Phi^\dagger}\mathrm{e}^{\mathsf{V}}\mathrm{e}^{\mathrm{i}\Phi} \iff \mathrm{e}^{-\mathrm{i}\Phi}\mathrm{e}^{-\mathsf{V}}\mathrm{e}^{\mathrm{i}\Phi^\dagger} , \tag{3.104}$$

  where Φ is a chiral superfield. [To first order in Φ this constitutes (3.96). See, e.g., Chap. 7 in either [11] or [47].] The reasoning here is that $\left(\mathrm{e}^{\mathsf{V}}\right)' = \mathrm{e}^{-\mathrm{i}\Phi^\dagger}\mathrm{e}^{\mathsf{V}}\mathrm{e}^{\mathrm{i}\Phi}$ is required for $\Phi^\dagger\mathrm{e}^{\mathsf{V}_{(a)}\mathcal{T}^{(a)}}\Phi$ to be gauge invariant [1, 2, 7, 13].
- We now adopt the Wess–Zumino gauge, whence

$$\mathsf{F}_A^{(a)} = -\mathrm{i}\lambda_A^{(a)}(y) + \theta_A\mathsf{d}^{(a)}(y) + \mathrm{i}(\sigma^{\mu\nu}\theta)_A f_{\mu\nu}^{(a)}(y) + \theta\theta\left(\sigma^m D_m\bar{\lambda}^{(a)}(y)\right)_A , \tag{3.105}$$

  where

$$f_{\mu\nu}^{(a)} = \partial_\mu v_\nu^{(a)} - \partial_\nu v_\mu^{(a)} - \frac{1}{2}\check{f}^{(a)(b)(c)}v_\mu^{(b)}v_\nu^{(c)} , \tag{3.106}$$

  with the gauge covariant derivative $\tilde{D}_\mu$ given by

$$\tilde{D}_\mu \overline{\lambda}^{(a)} \equiv \partial_\mu \overline{\lambda}^{(a)} - \check{f}^{(a)(b)(c)} v_\mu^{(b)} \overline{\lambda}^{(c)} . \tag{3.107}$$

For the simpler case, the method yields

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &\sim \int d^2\theta \text{Tr}\, \mathsf{F}^A \mathsf{F}_A \\ &\equiv \text{Tr}\left(-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \mathrm{i}\overline{\lambda}\overline{\sigma}^i \tilde{D}_i \lambda + \frac{1}{2}\mathsf{d}^2\right) . \end{aligned} \tag{3.108}$$

3.4 $N = 1$ Supergravity

Now we move on to our main concern, namely, the basic elements of *supergravity* (SUGRA). SUGRA is *the* gauge theory of the spacetime symmetries *plus* SUSY. There is a *local* symmetry relating bosonic and fermionic fields, leading to a new field theory of gravity. In fact, SUGRA can only be implemented if spacetime is curved. Gravity is an essential constituent.

Note 3.12 The reader may be wondering exactly *how* gravity (or general relativity) can be retrieved. A full and thorough explanation would require a whole book on this subject alone. However, a simplified manner to establish this is through the algebra (3.6) (or (3.17), (3.18)), and by looking at

$$\left[\delta_{\varepsilon 1}(x^\nu), \delta_{\varepsilon 2}(x^\nu)\right]\phi = -\mathrm{i}\left[\varepsilon_1(x^\nu)\sigma^\nu \varepsilon_2^\dagger(x^\nu) - \varepsilon_2(x^\nu)\sigma^\nu \varepsilon_1^\dagger(x^\nu)\right]\partial_\nu \phi .$$

Global SUSY induces spacetime translations, and *if* the SUSY transformations become local, then ‘translations’ that vary from point to point in spacetime will occur, i.e., generating general coordinate transformations in spacetime. This means a geometric theory of spacetime, i.e., a theory (including properties from general relativity) for the gravitational field. In other words, by gauging SUSY, a geometrodynamical spacetime emerges.

The richness goes beyond the feature described in Note 3.12. In addition, there will be a fermionic partner of the graviton, namely the spin 3/2 *gravitino* field. Overall, this establishes a *new* theory of gravitation, which includes general relativity and general coordinate transformations as a bosonic limit. The new fermionic fields only introduce changes at very short distances and hence very high energies. At the level of classical predictions, general relativity will emerge as the theory to employ.

The question at this stage is *how* to produce such a SUGRA theory. Plenty of detailed descriptions can be found in the literature. Herein, we will mainly follow the Noether procedure [1, 20], subsequently pointing to some features implied by SUGRA, whose domains will be explored from a Hamiltonian perspective (hence emphasizing the corresponding symmetries) in the next chapter.

To produce the action for pure (i.e., no matter yet) $N = 1$ SUGRA, we must first establish a Lagrangian for the supergravity multiplet (2, 3/2) that is invariant under *global* SUSY transformations. The Noether procedure will then lead, in an iterative method, using the associated conserved currents of the global case, to the *locally* SUSY Lagrangian for the supergravity multiplet (2, 3/2).

3.4.1 Noether Method

A Lagrangian for the supergravity multiplet (2, 3/2) that is invariant under *global* SUSY transformations can be considered on the basis of the SUSY transformations and the Wess–Zumino model discussed in Sect. 3.2.3. On-shell, we have a sum of a quadratic kinetic term for the scalar field and a linear term (in the derivatives) for the fermion. Therefore, let us start, adapting from [1], with a Lagrangian containing a quadratic kinetic term in the tetrad (or metric $g_{\mu\nu}$) and a linear derivative term for the gravitino $\psi_\mu^{[a]}$ (see Appendix A for the notation). This means taking for the (massless) gravitino $\psi_\mu^{[a]}$ the Rarita–Schwinger action in 4-component spinor notation [1, 20, 47]

$$S_{\text{RS}}^{\text{linear}} \simeq \frac{1}{2}\mathrm{i}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma\,. \tag{3.109}$$

But for the graviton, instead of the Einstein–Hilbert action of pure gravity, we take a linearized version [1, 20] (often employed within the context of, e.g., gravitational waves), where the graviton (tetrad) term is quadratic[10]:

$$S_{\text{EH}}^{\text{linear}} \simeq \frac{1}{2\mathsf{k}^2}\int \mathrm{d}^4x \left(R_{\mu\nu}^{\text{linear}} - \frac{1}{2}\eta_{\mu\nu}R^{\text{linear}}\right)h^{\mu\nu}\,, \tag{3.110}$$

where

$$R_{\mu\nu}^{\text{linear}} \equiv \frac{1}{2}\left(-\frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda\partial x_\lambda} + \frac{\partial^2 h_\nu^\lambda}{\partial x^\mu\partial x^\lambda} + \frac{\partial^2 h_\mu^\lambda}{\partial x^\nu\partial x^\lambda} - \frac{\partial^2 h_\lambda^\lambda}{\partial x^\mu\partial x^\nu}\right), \tag{3.111}$$

$$R^{\text{linear}} \equiv \eta^{\mu\nu}R_{\mu\nu}^{\text{linear}}, \tag{3.112}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathsf{k}h_{\mu\nu}\,. \tag{3.113}$$

The factor k in (3.113) is introduced so that $h_{\mu\nu}$ has unit mass dimension, as appropriate to a bosonic graviton [1]. Note that $S_{\text{RS}}^{\text{linear}}$ and $S_{\text{EH}}^{\text{linear}}$ are invariant under gauge

[10] In the full theory of general relativity, it is not even polynomial.

transformations, $\varepsilon^{[a]}$ is an arbitrary Majorana spinor parameter, and ξ is an arbitrary 4-vector parameter, where

$$\psi_\mu \longrightarrow \psi_\mu + \delta_\varepsilon \psi_\mu \equiv \psi_\mu + \partial_\varepsilon \psi_\mu \,, \tag{3.114}$$

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \delta_\xi h_{\mu\nu} \equiv h_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \,. \tag{3.115}$$

It then turns out that the action $S = S_{\rm RS}^{\rm linear} + S_{\rm EH}^{\rm linear}$ is (on-shell) globally SUSY invariant, up to the gauge transformations (3.114) and (3.115), with the SUSY transformations associated with the *global* SUSY change of $\exp(\varepsilon Q)$:

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \delta_\varepsilon h_{\mu\nu} \equiv h_{\mu\nu} - \frac{\mathrm{i}}{2}\overline{\varepsilon}\left(\gamma_\mu \psi_\nu - \gamma_\nu \psi_\mu\right) \,, \tag{3.116}$$

$$\psi_\mu \longrightarrow \psi_\mu + \delta_\varepsilon \psi_\mu \equiv \psi_\mu - \mathrm{i}\sigma^{\rho\tau} \partial_\rho h_{\tau\mu} \varepsilon \,. \tag{3.117}$$

Here, ε and Q are Majorana spinors and Q is a SUSY 'generator'. The equations of motion are

$$R_{\mu\nu}^{\rm linear} = 0 \,, \tag{3.118}$$

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma = 0 \iff \gamma^\mu \left(\partial_\mu \psi_\nu - \partial_\nu \psi_\mu\right) = 0 \,. \tag{3.119}$$

Therefore, the algebra (3.114), (3.115), (3.116), (3.117) of gauge and SUSY transformations[11] closes, i.e., taken together with spacetime translations *and* the equations of motion.

For the Noether procedure, we shall now consider instead a *local* SUSY transformation $\exp[\mathrm{i}\overline{\varepsilon}(x) Q]$, with ε and Q Majorana spinors that now depend on the point of spacetime. With (3.116) and (3.117) changed by $\varepsilon \to \varepsilon(x)$, the action $S = S_{\rm RS}^{\rm linear} + S_{\rm EH}^{\rm linear}$ changes by

$$\delta S \simeq \int \mathrm{d}^4 x \overline{J}^\mu \partial_\mu \varepsilon \,, \tag{3.120}$$

$$\overline{J}^\mu \simeq \frac{\mathrm{i}}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \sigma^{\lambda\tau} \partial_\tau h_{\tau\sigma} \,, \tag{3.121}$$

where $\overline{J}^\mu$ is the (Majorana) vector–spinor Noether current. We now construct

$$S' \equiv S_{\rm RS}^{\rm linear} + S_{\rm EH}^{\rm linear} - \frac{\mathsf{k}}{4} \int \mathrm{d}^4 x \overline{J}^\mu \psi_\mu \,, \tag{3.122}$$

which is invariant up to order k under the transformation

[11] Where $\varepsilon_\mu \equiv -\mathrm{i}\overline{\varepsilon}\gamma^\tau \varepsilon h_{\tau\mu}$.

$$\psi_\mu \longrightarrow \psi_\mu + \delta_\varepsilon \psi_\mu \equiv \psi_\mu - \mathrm{i}\sigma^{\rho\tau}\partial_\rho h_{\tau\mu}\varepsilon + \frac{2}{\mathsf{k}}\partial_\mu \varepsilon\,. \tag{3.123}$$

Proceeding in this manner, the action for $N = 1$ SUGRA (where $N = 1$ refers to the number of gravitinos) is found to be[12]

$$S^{N=1}_{\mathrm{SUGRA}} \simeq -\frac{1}{2\mathsf{k}^2}\int \mathrm{d}^4x \left|\det e^a_\mu\right| R - \frac{1}{2}\mathrm{i}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma\,, \tag{3.124}$$

which is invariant to *all* orders in k under the local SUSY transformations

$$e^a{}_\mu \longrightarrow e^a{}_\mu + \delta_\varepsilon e^a{}_\mu \equiv e^a{}_\mu - \mathrm{i}\mathsf{k}\overline{\varepsilon}\gamma^a\psi_\mu\,, \tag{3.125}$$

$$\psi_\mu \longrightarrow \psi_\mu + \delta_\varepsilon \psi_\mu \equiv \psi_\mu + \frac{2}{\mathsf{k}} D_\mu \varepsilon\,, \tag{3.126}$$

with the rather *particular* covariant derivative

$$D_\mu \equiv \partial_\mu - \mathrm{i}\omega_{\mu ab}\frac{\sigma^{ab}}{4}\,. \tag{3.127}$$

which generally differs from the usual minimal covariant derivative, viz.,

$$\mathbf{D}_\mu \psi_\sigma \equiv \partial_\mu \psi_\sigma - \mathrm{i}\omega_{\mu ab}\frac{\sigma^{ab}}{4}\psi_\sigma - \Gamma^{(0)\,\alpha}_{\mu\sigma}\psi_\alpha\,,$$

where $\Gamma^{(0)\,\alpha}_{\mu\sigma}$ is the standard Christoffel connection term of general relativity, since there is *no* Christoffel connection term ${}^{(0)}\Gamma^\alpha_{\rho\sigma}\psi_\alpha$ in (3.127). Moreover, notice that the *full* connection inducing torsion is (see Note 2.6 and Exercise 2.1)

$$\omega_{\mu ab} = \omega^{(0)}_{\mu ab} + \frac{\mathrm{i}\mathsf{k}^2}{4}\left(\overline{\psi}_\mu\gamma_a\psi_b + \overline{\psi}_a\gamma_\mu\psi_b - \overline{\psi}_\mu\gamma_b\psi_a\right)\,, \tag{3.128}$$

$$\omega^{(0)}_{\mu ab} = \frac{1}{2}e^\nu{}_a\left(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\right) + \frac{1}{2}e^\rho{}_a e^\sigma{}_b \partial_\sigma e_{\rho c} e_\mu{}^c - (a \leftrightarrow b)\,, \tag{3.129}$$

$$\kappa_{\mu ab} = \frac{\mathrm{i}\mathsf{k}^2}{4}\left(\overline{\psi}_\mu\gamma_a\psi_b + \overline{\psi}_a\gamma_\mu\psi_b - \overline{\psi}_\mu\gamma_b\psi_a\right)\,, \tag{3.130}$$

where $\omega^{(0)}_{\mu ab}$ is the standard spin connection of general relativity (*no* torsion) and $\kappa_{\mu ab}$ is the contorsion tensor (referring to torsion). In addition, we have for the curvature

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[12] Remarkably, it almost changes $S^{\mathrm{linear}}_{\mathrm{EH}}$ into the full Einstein–Hilbert Lagrangian S_{EH}, replacing the plain derivatives in $S^{\mathrm{linear}}_{\mathrm{RS}}$ by *specific* covariant derivatives. More precisely (for more details, see Sect. 3.4.2), we will have a non-minimal covariant derivative D_μ (*first* order formalism) or instead the usual covariant derivative [14–16, 18, 19], but with extra terms quartic in the gravitino in the action (*second* order formalism).

$$R_{\mu\nu ab} = \left(\partial_\mu \omega_{\nu ab} + \omega^c_{\mu a}\omega_{\nu cb}\right) - (\mu \leftrightarrow \nu) \,, \tag{3.131}$$

$$R = e^{a\mu} R_{\mu a} \,, \qquad R_{\mu a} = e^{b\nu} R_{\mu\nu ab} \,. \tag{3.132}$$

Note 3.13 For completeness, let us indicate some relevant expressions:

- The definition of the torsion $\xi^\rho{}_{\mu\nu}$, viz.,

$$\xi_{\mu\nu}{}^\rho = \frac{1}{2}\left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right) \,. \tag{3.133}$$

- The torsion can be related to $\omega_{\nu ab}$, by means of the metric–vierbein postulate $\mathbf{D}_\mu g_{\alpha\beta} = 0 = \mathbf{D}_\mu e_\nu{}^a$ and (3.128), (3.129), and (3.130), which follow from the equations of motion for $\omega_{\nu ab}$ and for the *usual* Christoffel symbol $\Gamma^{(0)\rho}_{\mu\nu}$ in terms of the metric alone:

$$\Gamma_{\mu\nu}{}^\rho = \Gamma^{(0)\rho}_{\mu\nu} - \kappa_{\mu\nu}{}^\rho \,, \tag{3.134}$$

$$\kappa_{\mu\nu\rho} = -\xi_{\mu\nu\rho} + \xi_{\nu\rho\mu} - \xi_{\rho\mu\nu} \,. \tag{3.135}$$

- The explicit expression for the torsion in terms of the gravitino field (see Sect. 3.4.3):

$$\xi_{\mu\nu\rho} \equiv \frac{\mathrm{i}}{2}\mathsf{k}^2 \overline{\psi}_\mu \gamma_\rho \psi_\nu \,. \tag{3.136}$$

Note 3.14 On the grounds of full SUSY invariance, the action (3.124) can be *extended*. It is invariant under local SUSY, Lorentz, and general coordinate transformations. However, we have mentioned (see discussion of the Wess–Zumino model in Sect. 3.2.3) that *auxiliary* fields may be of use to close the full algebra of transformations in the case of global SUSY (otherwise, it closes only on-shell). Auxiliary fields are also of relevance in (a) the presence of interactions, allowing the SUSY transformations *not to depend* on the particular potential of the model (Sect. 3.2.3), and (b) in the corresponding quantum theory [20, 47].

So what is the procedure when considering *local* SUSY transformations? The reader should notice that the same arguments apply. On-shell, both the vierbein (tetrad) and the gravitino have two effective degrees of freedom. Off-shell $e_\mu{}^a$ corresponds to 10 (= 16 − 6 from the local Lorentz group) degrees of freedom and $\overline{\psi}^{[a]}_\sigma$ to 16. Hence, six more bosonic degrees of freedom could be added off-shell. This can be done in the form of an (axial) vector field A_μ, a scalar field s, and a pseudoscalar field p. The action then becomes (see also [20, 47, 56])

$$S^{N=1}_{\text{SUGRA}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x \left|\det e_\mu{}^a\right| R - \frac{1}{2}\mathrm{i}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma + \frac{1}{3}\int \mathrm{d}^4x \left|\det e_\mu{}^a\right| \left(s^2 + p^2 - A_\mu^2\right) , \quad (3.137)$$

with the equations of motion for auxiliary fields being simply $s = p = A_\mu = 0$. The algebra of transformations closes off-shell as required, with the gauge transformation rules being the same.

3.4.2 First versus Second Order Formalism

There are now a few points to elaborate on. Some will be rather lengthy, but they will help us to address some questions the reader may already have in mind.

How Can the Noether Method Be Summarized?

This method allows one to retrieve an action S' with *local* symmetry, given an action S invariant under a *global* symmetry $\varphi \to \mathrm{e}^{-\mathrm{i}\varepsilon}\varphi$. If $S[\varphi]$ and $\varphi \to \mathrm{e}^{-\mathrm{i}\varepsilon(x)}\varphi$, then we use $\delta S = \int \mathrm{d}^4x\, J^\mu \partial_\mu \varepsilon$, J^μ being the Noether current associated with $\varphi \to \mathrm{e}^{-\mathrm{i}\varepsilon}\varphi$. To restore invariance, a term with the new gauge field A_μ is added to yield $S' = S - \int \mathrm{d}^4x\, J^\mu A_\mu$:

- For quantum electrodynamics, this is achieved in a single step. We simply add $\int \mathrm{d}^4x\, J^\mu A_\mu$ and there is no need here to change the local transformation [1].
- In other situations (e.g., Yang–Mills or $N = 1$ SUGRA above), the cancellation only occurs to some (higher) order in an expansion parameter. Geometrically speaking, an interaction is required for higher orders, until a complete invariance at all orders is retrieved. At each subsequent stage, a further term must be added to the action to cancel the variation (see (3.122)) *and* further terms may need to be added to the transformation of the gauge field introduced (see (3.123)), so that the algebra closes.

Was SUGRA Originally Derived in the Manner Summarized Above?

The fundamental references [14–16, 18, 19, 39, 20] indicate a somewhat different evolution. The Noether procedure is a compact method, once we have found out what to obtain and how to obtain it. It expresses the idea that SUGRA is the gauge theory of the super-Poincaré algebra [20, 47, 56]. SUGRA was originally developed via the *first-order* and *second-order* formulations[13]:

- In the first-order formalism, the vierbein (tetrad), the gravitino, *and* the spin connection $\omega_{\mu ab}$ are treated as *in*dependent variables and we eventually get $\omega_{\mu ab}(e,\psi)$ from the equation of motion. The first-order action $S^{(\mathbf{1})\,N=1}_{\text{SUGRA}}$, invariant under SUSY transformations $\delta^{(\mathbf{1})}e_\mu{}^a$, $\delta^{(\mathbf{1})}\psi_\sigma^{[a]}$, $\delta^{(\mathbf{1})}\omega_{\mu ab}$ [20], takes the form

$$S^{(\mathbf{1})\,N=1}_{\text{SUGRA}} = \frac{1}{2\mathsf{k}^2}\int \mathrm{d}^4x\,\left|\det e_\mu{}^a\right| R\,\left(e_\mu{}^a,\omega_{\mu ab}\right) - \frac{1}{2}\mathrm{i}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma\,, \tag{3.138}$$

with the definitions (3.127), (3.131), and (3.132) (see also Appendix A). Variation of (3.138) with respect to $e_\mu{}^a$, ψ_σ, and $\omega_{\mu ab}$ provides the equations of motion:

$$R^\mu_{\ a} - \frac{1}{2}e_\mu{}^a R = -\mathsf{k}^2 e^{-1}\varepsilon^{\nu\mu\rho\sigma}\left(\overline{\psi}_\nu\gamma_5\gamma_a D_\rho\psi_\sigma\right)\,, \tag{3.139}$$

$$0 = \varepsilon^{\mu\nu\rho\sigma}\left(\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{1}{2}\gamma_5\gamma_\lambda\xi^\lambda_{\nu\rho}\psi_\sigma\right)\,, \tag{3.140}$$

$$\omega_{\mu ab} = \omega^{(0)}_{\mu ab} + \kappa_{\mu ab}\,, \tag{3.141}$$

with (3.129) and (3.133), (3.134), (3.135), and (3.136):

$$\xi^\rho_{\mu\nu} = \frac{1}{2}\left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right)\,, \tag{3.142}$$

$$\Gamma^\rho_{\mu\nu} = \Gamma^{(0)\rho}{}_{\mu\nu} - \kappa^\rho_{\mu\nu}\,. \tag{3.143}$$

The relation between torsion and the spin connection (3.141), (3.129), with (3.133), (3.134), is obtained from the metric postulate $g_{\rho\sigma;\mu} = 0$ and the vierbein equivalent $e_a{}^{\nu;\mu} = 0$, where, e.g., $\mathbf{D}_\mu\psi_\sigma = D_\mu\psi_\sigma - \Gamma^\alpha_{\mu\sigma}\psi_\alpha$.
- In the second-order formalism, the vierbein (tetrad) and the gravitino are the independent variables. The corresponding action $S^{(\mathbf{2})\,N=1}_{\text{SUGRA}}$ is obtained by substituting the equation of motion for $\omega_{\mu ab}(e,\psi)$ obtained from the first-order action $S^{(\mathbf{1})\,N=1}_{\text{SUGRA}}$, leading to an action invariant under SUSY transformations $\delta^{(\mathbf{2})}e_\mu{}^a$, $\delta^{(\mathbf{2})}\psi_\sigma$ obtained from $\delta^{(\mathbf{1})}e_\mu{}^a$, $\delta^{(\mathbf{1})}\psi_\sigma$, with substitution of $\omega_{\mu ab}(e,\psi)$ retrieved from $S^{(\mathbf{1})\,N=1}_{\text{SUGRA}}$.

[13] An alternative is the 1.5 formalism, which is an 'educated' simplification [20]. It combines the virtues of both the first and second order formalisms. In essence, for an action with $S(e,\psi,\omega[e,\psi])$, we only vary it for e and ψ, putting $\delta\omega = 0$ wherever necessary, e.g., for any complicated expression when applying the chain rule to $\omega[e,\psi]$. Note that $\delta\omega = 0$ is also obtained when gauging the super-Poincaré algebra.

- The immediate difference between the first-order action $S^{(\mathbf{1})\,N=1}_{\mathrm{SUGRA}}$ and the second-order action $S^{(\mathbf{2})\,N=1}_{\mathrm{SUGRA}}$ is the non-minimal (not quite fully covariant derivative) D_μ (see (3.127)) and the subsequent presence of quartic gravitino terms in $S^{(\mathbf{2})\,N=1}_{\mathrm{SUGRA}}$.

3.4.3 SUGRA and Theories of Gravitation with Torsion

One question the reader has probably been formulating is this: If (3.124) represents a new theory for the gravitational interaction, what does that mean with regard to the properties and structure of a corresponding and obviously new and different spacetime?

It should be noted that the spacetime of $N = 1$ SUGRA lies in the class of Riemann–Cartan spaces [60, 61]. These are a particular type of *affinely connected* metric space from which Riemann spaces of the kind used in general relativity are obtained when there is no torsion. In slightly more detail [39]:

- In an affinely connected space (manifold), we have[14] an affine connection $\breve{\Gamma}$, with which we define a covariant derivative $\breve{D}$ that transforms as a tensor under corresponding general coordinate transformations. The affine connection structure allows one to define parallel transport, and hence provides a procedure for comparing vectors at different points in the manifold, the difference being indicated through the covariant derivative. (A notion of self-parallel curve follows if the derivative is zero.) The antisymmetric part of $\breve{\Gamma}$ is the *torsion* (see (3.133)):
  - If the torsion is *non*-zero, this means that parallel transport along vectors v_1 and v_2 as well as in the reverse order (i.e., v_2 then v_1) forms an infinitesimal parallelogram that does *not* close. The measure of the non-closure is given by the torsion.
  - Curvature involves parallel transporting a vector u along a closed curve from P to Q, first going along v_1 then v_2, and then comparing it with the result of going the other way (v_2 then v_1). The measure of the difference is given by the Riemann curvature tensor. As one would expect, there is a relation between the curvature and torsion.
  - A metric $\breve{g}$ can be defined in the tangent space of the manifold, allowing the definition of inner products between vectors and a concept of length. With a metric, we can define the length of a curve and establish geodesic equations for curves of minimal (maximal) length.[15]
  - Hence, we can have *two independent fields* on a manifold: the affine connection $\breve{\Gamma}$ and the metric $\breve{g}$.

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[14] The breve on symbols indicates a quantity in a theory that is still more generic than general relativity. Eventually, the whole setting will converge to the usual quantities within general relativity.

[15] In fact, this is an alternative way of defining the concept of parallelism, which may or may not coincide with the above (depending on whether there is torsion).

– Then, we can *impose* the metric postulate $\mathbf{D}_\mu \breve{g}_{\nu\rho} = 0$, with a view to obtaining the affine connection as a function of the metric, eventually getting $\Gamma^{(0)}{}^{\rho}_{\mu\nu}$ in terms of a metric g, as in general relativity (this is the Levi-Civita connection, with components known as the Christoffel symbols). In this framework, we have a *particular* affinely connected metric space: the Riemann–Cartan space. The point to notice is that the torsion can still be present, not defined in terms of the metric. In a Riemann–Cartan space we have (3.134) and (3.135).
– Requiring the torsion to vanish, so that $\breve{\Gamma}$ reduces to $\Gamma^{(0)}{}^{\rho}_{\mu\nu}$, a Riemannian space, as used in general relativity, can be extracted.

- We now consider a wider context for the affinely connected metric space, and one that is more suitable for dealing with fermions, i.e., spinors [62]. Instead of a coordinate basis[16] $\hat{\mathbf{e}}_\mu$, we can take $\mathsf{e}_a = e^\mu{}_a \hat{\mathbf{e}}_\mu$ in the tangent space, relating world indices (μ) and the tangent space (a):

– The metric components can now be expressed as $\eta_{ab} \equiv g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}$.
– In the context of this basis, a covariant derivative such as

$$e^\mu{}_a \mathbf{D}_\mu \vartheta^b = \mathbf{D}_a \vartheta^b = \partial_a \vartheta^b + \breve{\omega}^b_{ac} \vartheta^c \,,$$

can be introduced to transform tangent space objects, associated with a *new* connection ω.
– Then a suitable curvature tensor (see (3.131)) is defined by

$$\breve{R}_{\mu\nu ab} = \left(\partial_\mu \breve{\omega}_{\nu ab} + \breve{\omega}^c_{\mu a} \breve{\omega}_{\nu cb}\right) - (\mu \leftrightarrow \nu) \ .$$

– In manner similar to above, we can introduce other postulates, in order to relate $\breve{\omega}$, $\breve{\Gamma}$, and e (and $\breve{g}$, indirectly). We obtain the relation between connections[17]

$$\omega^b_{\mu a} = \Gamma^b_{\mu a} - e^\nu_a \partial_\mu e^b_\nu \,,$$

by imposing the first vierbein postulate (which allows $e^a_\nu \mathbf{D}_\mu v^\nu = \mathbf{D}_\mu v^a$)

$$\mathbf{D}_\mu e^\nu_a \equiv \partial_\mu e^\nu_a - e^\nu_b \omega^b_{\mu a} + \Gamma^\nu_{\mu\rho} e^\rho_a = 0 \,.$$

– Imposing the metric postulate in the vierbein context (the second vierbein postulate), we retrieve a Riemann–Cartan space (see above), and we have only one independent connection, but Γ is generic in the sense that $\Gamma^\rho_{\mu\nu} = \Gamma^{(0)}{}^{\rho}_{\mu\nu} - \kappa_{\mu\nu}{}^\rho$, and similarly for ω, related to Γ by $\omega^b_{\mu a} = \Gamma^b_{\mu a} - e^\nu_a \partial_\mu e^b_\nu$, with

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[16] Meaning $\left[\hat{\mathbf{e}}_\mu, \hat{\mathbf{e}}_\nu\right] = 0$.

[17] Henceforth, dropping the breve on each symbol, we can write

$$R_{\mu\nu\rho\sigma}(\Gamma) = e^a_\rho e^\sigma_b R^b_{\mu\nu a}(\omega) \,, \qquad \xi^a_{\mu\nu} = 2\left(\omega^a_{[\mu|\nu]} - \partial_{[\mu} e^a_{\nu]}\right) \,.$$

$$\omega^c_{ab} = \omega^c_{ab}(e) + \kappa^c_{ab} \,,$$

and $\omega^c_{ab}(e)$ related[18] to $\Gamma^{(0)\rho}_{\mu\nu}(g)$ and given by[19]

$$\omega^c_{ab}(e) = \Gamma^c_{ab}(e) - \Omega^c{}_{ab} + \Omega_b{}^c{}_a - \Omega_{ab}{}^c \,,$$
$$\Omega_{ab}{}^c = e^\mu{}_a e^\nu{}_b \partial_{[\mu} e^c_{\nu]} \longleftarrow [e_a, e_b] = -2\Omega_{ab}{}^c e_c \,.$$

- As an aside, with an *orthonormal* basis $g_{ab} = \eta_{ab}$, where

$$\Gamma^c_{ab}(e) = \frac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right) = 0 \,,$$
$$\omega^c_{ab} = \omega^c_{ab}(e) + \kappa^c_{ab} \,,$$
$$\omega^c_{ab}(e) = -\Omega^{\;\;c}_{ab} + \Omega^{\;c}_{b\;a} - \Omega^{\;\;c}_{ab,} \,,$$

  and restricting to transformations preserving the metric g_{ab}, we can define a *gauge* theory suitable for representations of spinors. In other words, the theory is invariant under a local Lorentz group.
- A (gauge) connection $\omega^\mu_{ab} = \omega^\mu_{[ab]}$ thus follows. It is selected by $\mathbf{D}g_{ab} = 0$ and the properties of the gauge indices ab identify it as the spin connection. Then we have a covariant derivative of spinors, for example.
- The above formalism introducing Riemann–Cartan spaces is thus suitable for including and dealing with fermions (spinors). Coupling to torsion, we recover the Cartan–Sciama–Kibble (CSK) theory [39]. Relating the spin connection to the vierbein (tetrad) by means of the (first) tetrad postulate[20] will allow us to recover general relativity (the metric g being the only field), but at a price: the connection will *not* be a dynamical field, in contrast to what happens in Yang–Mills theories.
- As we approach the end (and actually return to the starting point!) of this digression on (3.124)–(3.137), let us indicate how torsion can be dealt with (i.e., within the CSK theory [39]) and hence, arrive (for a canonical point of view) 'at the gates' of $N = 1$ SUGRA, which will be analysed in detail in Chap. 4 with the aim of exploring the dynamics of the early universe (see Part III):

  – The basic approach is to take the torsion $\xi_{\mu\nu\rho}$ as a *new* field, whose equation of motion is *algebraic* and then give $\xi_{\mu\nu\rho}$ as a function of the other fields. The action would take the form

$$S_{\text{CSK}}\left[e^\nu_a, \xi_{\mu\nu\rho}\right] \sim \int \mathrm{d}^4x \left|\det e^a_\mu\right| R\left(e^a_\mu, \xi_{\mu ab}\right) \,,$$

  where the Ricci curvature depends on the tetrad *and* the torsion terms.

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[18] Similarly to $\omega^b_{\mu a} = \Gamma^b_{\mu a} - e^\nu_a \partial_\mu e^b_\nu$.

[19] Requiring zero torsion, we get the usual Levi-Civita connection (Christoffel symbols) $\Gamma^{(0)\rho}_{\mu\nu}(g)$ and the Cartan connection $\omega^c_{ab}(e)$, suitable for a Riemannian spacetime.

[20] $\mathbf{D}_\mu e^\nu_a = 0$ and $\omega^b_{\mu a} = \Gamma^b_{\mu a} - e^\nu_a \partial_\mu e^b_\nu$.

– The equations of motion, assuming the presence of a matter Lagrangian, are of the form [39]

$$G^{(\alpha\beta)} - f_1\left[\xi^{\mu(\alpha\beta)}\right] \sim M^2_{\text{Planck}}\mathsf{T}^{(\alpha\beta)}\,,$$
$$f_2\left[\xi^{\mu(\alpha\beta)}\right] \sim \Pi^{\mu(\alpha\beta)}\,,$$

where $G^{(\alpha\beta)}$ is the symmetric part of the Einstein tensor, $f_1\left[\xi^{\mu(\alpha\beta)}\right]$ is a function of the torsion obtained from a suitable covariant derivative, $f_2\left[\xi^{\mu(\alpha\beta)}\right]$ is a linear combination of torsion terms, and the *spin energy potential* is

$$\Pi^{\mu(\alpha\beta)} \equiv \frac{\delta L_{\text{matter}}}{\delta\xi^{\mu(\alpha\beta)}}\,,$$

similarly to the energy–momentum tensor

$$\mathsf{T}^{(\alpha\beta)} \equiv e^{\alpha}_{b}\frac{\delta L_{\text{matter}}}{\delta e^{\beta}_{b}}\,.$$

The interesting feature is that these equations can be assembled into one, viz., $G^{\alpha\beta}\left[\Gamma(g)\right] \sim \mathsf{T}^{\alpha\beta} + \mathcal{O}\left(M^4_{\text{Planck}}\right)$, indicating that corrections can be relevant:

1. To be more precise [39], the $\mathcal{O}\left(M^4_{\text{Planck}}\right)$ term comes from the use of a function $f_3\left[\xi^{\mu(\alpha\beta)}\right]$ quadratic in the torsion, where this is related to $\xi^{\mu\alpha\beta} \sim M^2_{\text{Planck}}\mathsf{S}^{\mu\alpha\beta}$, with $S^{\mu\alpha\beta}$ the *spin angular momentum*, and $S^{\mu ab}$ also admits a relation with $\partial L_{\text{matter}}/\partial\omega^{\mu}_{ab}$. So the matter spin energy potential is the origin of torsion.
2. The $\mathcal{O}\left(M^4_{\text{Planck}}\right)$ term is related to the density of (intrinsic) spins. At a quantum level, it predicts interactions between fermions (spinors), induced from the quadratic torsion terms as well as the covariant derivatives in L_{matter}. However, at large, currently observed scales $\mathcal{O}\left(M^2_{\text{Planck}}\right)$, it would be negligible.

– In essence, taking a CSK gravity theory coupled with a massless Rarita–Schwinger fermion $\psi^{[a]}_{\mu}$, so that torsion is induced by the gravitino, one obtains the geometrical structure of $N = 1$ SUGRA in four spacetime dimensions!

3.5 $N = 1$ Supergravity and Matter

Before proceeding with the Hamiltonian analysis of $N = 1$ SUGRA, it is important to consider how one can include matter in the theory, while complying with the requirements of invariance under local SUSY, local Lorentz, and generic coordinate transformations. On the one hand, for any SUGRA to make contact with reality, it has to be coupled to matter Lagrangians where the standard model constituents are present.[21] On the other hand, the consequences of having (super)matter have barely been explored within SQC and semiclassical SUGRA (see Chap. 4, Vol. II).

The easiest and most straightforward model to start with is the free massless chiral supermultiplet (Wess–Zumino) action in (3.15) with global SUSY. Requiring the action to be invariant under $\phi \to \phi + \delta\phi$, $\psi \to \psi + \delta\psi$, but taking $\delta\phi = \varepsilon^A \psi_A$, $\delta\psi_A = \mathrm{i}\left(\sigma^\nu \varepsilon^\dagger\right)_A \partial^\nu \phi$, such that[22] $\varepsilon = \varepsilon(x^\mu)$, we can follow Noether's procedure, i.e., (a) we add terms to the Wess–Zumino Lagrangian, and (b) we add terms to the SUSY transformations until invariance is retrieved.

More generally, within global SUSY, we can start from the Lagrangian for vector V and chiral superfields $(\Phi^\dagger, \Phi)$:

$$L_{\text{chiral + vector}} = \int \mathrm{d}^4\theta\, \mathsf{K}\left(\Phi^\dagger, \mathrm{e}^{2\zeta \mathsf{V}}\Phi\right) + \int \mathrm{d}^2\theta\, \mathsf{W}(\Phi) + \text{h.c.} + \int \mathrm{d}^2\theta\, \mathsf{f}_{(a)(b)}(\Phi) \mathsf{F}^A_{(a)} \mathsf{F}_{A(a)} + \text{h.c.}\,, \tag{3.144}$$

where $\mathsf{K}\left(\Phi^\dagger, \mathrm{e}^{2\zeta\mathsf{V}}\Phi\right)$ is a function which will allow a generic kinetic term for the scalar superfields, $\mathrm{e}^{2\zeta\mathsf{V}}$ is a coupling between the chiral supermultiplets and the vector field, $\mathsf{W}(\Phi)$ is a superpotential, $\mathsf{F}^A_{(a)}$ is the gauge field strength superfield (where A is a spinor index and (a) a gauge group index) constructed from V, and $\mathsf{f}_{(a)(b)}(\Phi)$ is a function of the chiral superfield (which can be $\delta_{(a)(b)}$ in suitable situations).

The use of the Noether method produces a rather long, but not particularly complicated Lagrangian, in which three blocks can be identified:

$$L = L_\mathrm{B} + L_\mathrm{F1} + L_\mathrm{F2}\,. \tag{3.145}$$

The first block L_B contains only bosonic fields, namely,

$$L_\mathrm{B} \simeq \frac{1}{\det e}\left\{R - \mathsf{G}^{\mathsf{I}}_{\mathsf{J}} \tilde{D}_\mu \phi_{\mathsf{I}} \tilde{\mathrm{D}}^\mu \phi^{\mathsf{J}*} + \mathrm{e}^{\mathsf{G}}\left[3 - \mathsf{G}_{\mathsf{I}}(\mathsf{G}^{-1})^{\mathsf{I}}_{\mathsf{J}}\mathsf{G}^{\mathsf{J}}\right]\right\} + \frac{1}{\det e}\left\{-\frac{1}{4}\left[\mathrm{Re}\, \mathsf{f}_{(a)(b)}\right] f_{(a)\mu\nu} f^{\mu\nu}_{(b)} + \frac{\mathrm{i}}{4}\left[\mathrm{Im}\, \mathsf{f}_{(a)(b)}\right] f_{(a)\mu\nu} \tilde{f}^{\mu\nu}_{(b)} - \frac{\zeta^2}{2}\left[\mathrm{Re}\, \mathsf{f}^{-1}_{(a)(b)}\right] \mathsf{G}^{\mathsf{I}} \mathsf{G}^{\mathsf{K}} \left[\mathcal{T}^{(a)}_{\mathsf{IJ}} \mathcal{T}^{(b)}_{\mathsf{KL}}\right] \phi_{\mathsf{J}} \phi_{\mathsf{L}}\right\}, \tag{3.146}$$

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[21] A vast set of publications is available [1, 8, 20, 47], and we will simply refer here to the basic procedures and pertinent elements that will be of use throughout Part III of this book.

[22] So the Majorana spinor $\varepsilon \to \begin{pmatrix} \varepsilon \\ \bar{\varepsilon} \end{pmatrix}$ is allowed to depend on the point of spacetime (see Sect. 3.3.3, Vol. II and Note 3.10).

where the second line corresponds to the inclusion of the vector supermultiplet elements, $\det e$ is the determinant of the vierbein e^a_μ, R is the usual curvature scalar, and the derivatives $\tilde{\mathrm{D}}_\mu$ are made covariant with respect to gravity (and also, in the usual way, to the gauge group *if* the vector supermultiplet is considered). Note that the SUGRA Lagrangian will depend on the function

$$\mathsf{G}(\phi^\dagger, \phi) \equiv \mathsf{I}(\phi^\dagger, \phi) + \ln |\mathsf{W}|^2 , \tag{3.147}$$

$$\mathsf{I}(\phi^\dagger, \phi) = -3 \ln \frac{\mathsf{K}(\phi^\dagger, \phi)}{3} , \tag{3.148}$$

with the function G now acquiring the designation of Kähler potential (despite the fact that K was employed as such in Sect. 3.3), satisfying the rather important transformation (which can be gauged)

$$\mathsf{I} \longrightarrow \mathsf{I} + \mathsf{h}(\phi) + \mathsf{h}^*(\phi^*) , \tag{3.149}$$

$$\mathsf{W} \longrightarrow \mathrm{e}^{-\mathsf{h}}\mathsf{W} . \tag{3.150}$$

It is defined for ϕ_I and its adjoint ϕ^{J*} by

$$\mathsf{G}^\mathsf{I} \equiv \frac{\partial \mathsf{G}}{\partial \phi_\mathsf{I}} , \qquad \mathsf{G}_\mathsf{J} \equiv \frac{\partial \mathsf{G}}{\partial \phi^{\mathsf{J}*}} , \qquad \mathsf{G}^\mathsf{I}_\mathsf{J} \equiv \frac{\partial^2 \mathsf{G}}{\partial \phi_\mathsf{I} \partial \phi^{\mathsf{J}*}} , \tag{3.151}$$

satisfying $(\mathsf{G}^{-1})^\mathsf{I}_\mathsf{J} \mathsf{G}^\mathsf{J}_\mathsf{K} = \delta^\mathsf{I}_\mathsf{K}$, and of course (see (3.106))

$$f_{(a)\mu\nu} \equiv \partial_\mu v_{\nu(a)} - \partial_\nu v_{\mu(a)} - \zeta \check{f}_{(a)(b)(c)} v_{\mu(b)} v_{\nu(c)} , \tag{3.152}$$

$$\tilde{f}_{(a)\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}_{(a)} . \tag{3.153}$$

Furthermore, note that:

- L_{F1} contains kinetic terms for the fermions, in particular, the Rarita–Schwinger term of $N = 1$ SUGRA, i.e., the action (3.124) is retrieved within (3.145). It also contains scalars coupled to fermions.
- L_{F2} includes terms quadratic in the fermions, namely the gravitino, coupled to functions such as $\mathrm{e}^{\mathsf{G}/2}$ of the scalars, and also quartic terms in the fermions.
- Writing $L_{\mathrm{F1}} + L_{\mathrm{F2}}$ out explicitly will not add significant information within the specific context of this book, and the reader can find them in the literature.
- For consistency, let us just indicate that the SUGRA action associated with (3.145) is invariant under the action of the local SUSY transformations on the fields e^a_μ, ψ^A_μ, φ, ψ, $v^{(a)}_\mu$, and λ_A (where the last two constitute the gauge vector and gaugino fields, respectively), generalizing those of (3.125) and (3.126).
- There are a few commonly made choices. For example, minimal kinetic terms, e.g., $\partial_\mu \varphi_\mathsf{I} \partial^\mu \varphi^*_\mathsf{I}$ or $\overline{\psi}_\mathsf{I} \gamma^\mu \partial_\mu \psi_\mathsf{I}$, are retrieved using

$$\mathsf{K}(\phi^\dagger, \phi) = -3 \exp\left(-\frac{\phi^\dagger \phi}{3}\right) , \tag{3.154}$$

which leads to

$$\mathsf{G}(\phi^\dagger, \phi) = \varphi_{\mathsf{I}} \varphi^{\mathsf{I}*} + \ln |\mathsf{W}|^2 , \tag{3.155}$$

and hence $\mathsf{G}^{\mathsf{I}}_{\mathsf{J}} = \delta^{\mathsf{I}}_{\mathsf{J}}$.

Summary and Review. With the canonical quantization of $N = 1$ SUGRA in sight (next chapter), here is a list of review points (within the context of this book!) concerning SUSY and SUGRA. The aim is to assist the committed explorer in assessing his or her progress:

1. Are SUSY and corresponding theories such as SUGRA or superstrings well motivated? What makes them so appealing to investigate (Sect. 3.1)?
2. How is it possible for spacetime coordinate transformations to be retrieved from the SUSY algebra (Sects. 3.2 and 3.4)?
3. Summarize the main features and extensions of the Wess–Zumino model for the chiral supermultiplet. Why are auxiliary fields so ‘relevant’ (Sect. 3.2.3)?
4. What is (SUSY) superspace and what are the essential (computational) benefits? How is the Wess–Zumino model retrieved (Sect. 3.3)?
5. What are the main features of the Kähler geometry and why is it relevant here (Sect. 3.3)?
6. Why is SUGRA an *extension* of general relativity (Sect. 3.4)?

Problems

3.1 SUSY Generators as Spin 1/2 Fermions

Investigate how and why a SUSY generator behaves as a spin 1/2 fermion under rotations.

3.2 Retrieving the Algebra (3.9), (3.10), and (3.11)

Determine and investigate how the algebra (3.9), (3.10), and (3.11) can be retrieved.

3.3 Spin Raising

Discuss how the spin can be raised or lowered by half a unit within the supermultiplet using Q and $Q^\dagger$.

3.4 Number of Bosonic and Fermion Degrees of Freedom

Explain why a supermultiplet always contains an equal number of bosonic and fermion degrees of freedom.

3.5 Massless and Massive Supermultiplets with *N*=1 or *N*=2 and with or without Central Charges

Investigate the massless and massive supermultiplets found in $N = 1$ and $N = 2$ SUSY, with or without central charges.

3.6 SUSY Invariant Lagrangians in Superspace and the Wess–Zumino Model

Analyse how SUSY invariant actions can be built from (SUSY) superspace and superfields, thereby retrieving the Wess–Zumino model as a special case.

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Chapter 4
Canonical Quantization of $N = 1$ Supergravity

This is where our journey of exploration will really begin. The last two chapters introduced the essential features of a quantum mechanical description of the early universe and the use of fermions in a supersymmetric setting:

- On the one hand, the analysis of general relativity from a Hamiltonian perspective [1–7] has brought new insights into the very origin of the universe[1] (as well as many other domains of research such as spacetime singularities, gravitationally driven chaos, primordial perturbations).
- On the other hand, SUGRA is overwhelmingly elegant, a fact quite manifest and subtly enriched in the recent context of superstring and *M*-theory [8–11]. More precisely, $N = 1$ SUGRA in 4D spacetime [12, 13] is a possible projection of the enticing theory of 11D $N = 1$ SUGRA, which constitutes one of the limit sectors of the (still not fully defined) *M*-theory, where superstring theory is also present. Moreover, as pointed out in the last chapter, 4D $N = 1$ SUGRA theory[2] includes Einstein's general relativity theory. Subsequently, superstring theory emerged as an immensely promising candidate for a unification theory, including gravity within a quantum mechanical setting.

Nevertheless, in spite of all of the above, an important viewpoint remained noticeably absent until the late 1980s and early 1990s. In fact, the revealing Hamiltonian formulation of cosmological dynamics and the elegant structure of SUGRA were displayed in a disjoint manner.

And this is manifestly insufficient. If SUSY plays a *fundamental* role in the evolution of the very early universe, a consistent quantum mechanical setting is required to involve *all* physical variables. In other words, a quantization scheme

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[1] In fact, we were given a methodology with which to probe the universe from a *non*-perturbative and quantum mechanical standpoint. Concepts such as the wave function and the creation of the universe, associated with a transition to classical states, were then extensively explored [14–21].

[2] Since the early 1970s, a considerable range of publications about SUSY and SUGRA can be found in the literature in the context of particle physics interactions. Fascinating results have been achieved [22, 12], notably finite values for specific interactions that were not renormalizable using just perturbative general relativity.

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(e.g., a canonical formulation) for SUGRA (and superstrings) is mandatory, from which specific cosmological applications ought to be retrieved.

A fruitful approach is to start with the extension of Hamiltonian techniques of analysis towards SUGRA, in a manner similar to general relativity. This is the topic of Sect. 4.1. In Sect. 4.2, we proceed to discuss the main features of the canonical quantization of SUGRA. In these sections we will be following in the pioneering footsteps of a whole host of authors [23–35].

We recall that, in Sect. 2.2.1 of Chap. 2, we mentioned briefly that we employ a $3 + 1$ decomposition of spacetime for general relativity. This process enabled us to identify the corresponding constraints, in the form of the Hamiltonian constraint (that leads to the Wheeler–DeWitt equation) and the diffeomorphism (or momentum) constraints, which also portray the invariances of the theory. From these constraints, wave functionals associated with specific models for the universe could be determined. A similar methodology and structure will emerge throughout this chapter, additionally enriched by the presence of (local) SUSY.

4.1 Hamiltonian Formulation

The description of a theory in Hamiltonian terms is of particular relevance. It allows us to identify a 'geometrical representation' and then use its underlying symmetries to various ends:

- To find specific classes of solutions, which is not obvious in the Lagrangian presentation.
- To establish the effective degrees of freedom of the theory
- To identify the conserved quantitities.

And even:

- To acquire a larger perspective wherein such theories can be located within others, something that is often concealed by a specific choice of coordinates.

In simpler contexts, invariance properties of any theory are most clearly accessible in a Hamiltonian formulation. In this sense, the Hamiltonian formulation is more fundamental and the quantum description should reflect this.

We pointed out in Chap. 2, and we will emphasize in this chapter as well for the case of $N = 1$ SUGRA [36, 37, 34, 38, 39, 40–44] that there is an interesting association between a particular spacetime geometry and the algebra of the transformation (deformation) generators of the hypersurfaces. This is quite elegant in general relativity and it will be particularly *fascinating*[3] to perform in $N = 1$

[3] In $N = 1$ SUGRA, we will find fourteen constraints associated with the invariances under generic changes of coordinates (four), local Lorentz rotations (six), and SUSY transformations (four). Moreover, hypersurface generators (of deformations) in the spacetime are obtained from the anticommutator of two (local) SUSY generators, hence widening the SUSY context, where a translation is retrieved from the anticommutation of two SUSY transformations (see Sect. 3.2).

SUGRA, because it may point the way to as yet uncharted directions. What are the physical consequences, in this context, of changing the usual Riemannian features of our spacetime?[4] One route of exploration is to enlarge the algebra of constraints (i.e., to include $N = 1$ SUGRA terms) and explore the changes in geometry and symmetry. The reason is as already indicated: the algebra constitutes an elegant means for defining the geometrical representation determined by the theory itself (see Sect. 2.6 and Note 2.12).

In addition, in a SUSY theory such as $N = 1$ SUGRA, we will see how the presence of spinor terms (see Appendix A) and fermionic features translates into characteristic geometrical properties through, e.g., the curvature tensor. In $N = 1$ SUGRA, the presence of the gravitino [on partnership terms with the tetrad (graviton)] will imply geometrical diferences.

4.1.1 Decomposition with Metric and Torsion

In order to build a solid and smooth transition from general relativity to $N = 1$ SUGRA, we will once again use a 4D general relativistic setting, but now enriched with torsion. Let us be more concrete [34]. For a $3 + 1$ decomposition of Einstein gravity with torsion, in particular, a Cartan–Sciama–Kibble (CSK) theory (see Sect. 3.4.3 and in particular [45]):

- The torsion ξ may be implemented through spinors and obtained *generically* from

$$\xi^{\lambda}_{\mu\nu} \equiv \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right) . \tag{4.1}$$

  This can be made more precise as follows. Take $\mathbf{e}_{\mu}$ as vectors in a coordinate basis, with $\mathbf{e}_i$ tangent to a spacelike hypersurface and the normal $\mathbf{n}$ with components $\mathbf{n} \to n_{\mu} = (-\mathcal{N}, 0, 0, 0)$. Then, with the components of the metric as in (2.6), (2.7), and (2.8), we can write

$$\mathbf{e}_{\mu;\nu} = \mathbf{e}_{\lambda}\Gamma^{\lambda}_{\mu\nu} , \tag{4.2}$$

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{(0)\,\lambda}_{\mu\nu} - \kappa_{\mu\nu}{}^{\lambda} , \tag{4.3}$$

$$\kappa_{\mu\nu}{}^{\lambda} = \xi_{\mu\nu}{}^{\lambda} - \xi_{\mu}{}^{\lambda}{}_{\nu} + \xi^{\lambda}{}_{\mu\nu} , \tag{4.4}$$

  where $\Gamma^{(0)\,\lambda}_{\mu\nu}$ are the usual Christoffel symbols[5] and $\kappa_{\mu\nu}{}^{\lambda}$ is the *contorsion* tensor.

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[4] For $N = 1$ SUGRA, we will have additional fermionic terms in the generators, compared with those in Chap. 2, as well as *new* generators. The geometry associated with $N = 1$ SUGRA (a new theory of gravitation, and with expected benefits within the quantum regimes) will be remarkably different from the Riemannian setting of general relativity.

[5] In this context, the Christoffel symbol in (2.9) is actually $\Gamma^{(0)\,\lambda}_{\mu\nu}$, as the reader has surely noticed [see also (4.3) and Sect. 3.4.3 for further explanation].

- The extrinsic curvature K_{ij} can *for this setting* be computed and retrieved from $\mathbf{n}_{;\mathbf{j}}$ using (see (2.9))

$$K_{ij} = -\mathcal{N}^{(4)}\Gamma^0_{ji} = \frac{1}{2\mathcal{N}}\left(-h_{ij,0} + \mathcal{N}_{i|j} + \mathcal{N}_{j|i}\right) - \kappa_{ij\perp} , \tag{4.5}$$

$$K_{(ij)} = \frac{1}{2\mathcal{N}}\left(-h_{ij,0} + \mathcal{N}_{i|j} + \mathcal{N}_{j|i}\right) + \tau_{(ij)} , \tag{4.6}$$

$$K_{[ij]} = \xi_{ji\perp} , \qquad \tau_{ij} \equiv 2\xi_{i\perp j} . \tag{4.7}$$

- We recall that we are using the notation of Sect. 2.2.2, viz.,

$$-A_\perp = A^\perp \equiv -n^\mu A_\mu = \mathcal{N} A^0 = -\frac{1}{\mathcal{N}}\left(A_0 - \mathcal{N}^i A_i\right) .$$

It is therefore instructive to compare (2.9) with (4.5), and in particular the presence of the last term $\kappa_{ij\perp}$. Notice now the form of (4.3) with its last term, whose complete expression is (4.4).

For general relativity including torsion, the gravitational dynamics involves the Einstein–Hilbert action (with a possible cosmological term Λ, and a matter sector) as in Sect. 2.2.2 and (2.10), (2.11). In more specific terms, the action (2.10), (2.11) must be within the framework of the 3 + 1 ADM decomposition split including torsion. Similarly to the general relativistic setting:

- ${}^{(4)}R = {}^{(3)}R - K_{ij}K^{ij} + K^2 - 2\,{}^{(4)}R_\perp{}^\alpha{}_{\perp\alpha}$,
- ${}^{(4)}R_\perp{}^\alpha{}_{\perp\alpha} = \left(n^\gamma n^\beta{}_{\wr\alpha} - n^\beta n^\gamma{}_{\wr\gamma}\right)_{\wr\beta} - K^{ij}K_{ij} + K^2 - 2\xi_\perp{}^{\alpha\beta} n_{\alpha\wr\beta}$,
- $K = -n^\gamma{}_{\wr\gamma}$,
- $K^{ij}K_{ji} = n^\beta{}_{\wr\gamma} n^\gamma{}_{\wr\beta}$,

where the symbol $\wr$ denotes covariant differentiation with respect to the full space-time metric including the spin connection. Consequently, the action becomes

$$\begin{aligned} S &\equiv \int_{\mathcal{M}} \mathrm{d}t\, L \\ &= \frac{1}{2\mathrm{k}^2}\int_{\mathcal{M}} \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{N}\sqrt{h}\Big\{{}^{(3)}R - 2\Lambda + \left[K_{(ij)} - 2\tau_{(ij)}\right]K^{(ij)} - K(K - 2\tau) \\ &\quad - K_{[ij]}K^{[ij]} + 2\tau_{[ij]}K^{[ij]} - q^i\rho_i + 2\rho^i_{||\,i} - 2\rho^i\rho_i\Big\} + S_{\mathrm{matter}} , \end{aligned} \tag{4.8}$$

where $\rho_i \equiv 2\xi_{ij}{}^j$, $q_i \equiv 4\xi_{i\perp}{}^\perp$ (see Appendix A). The canonical momenta are, again with a scalar field ϕ for matter sector,

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{2\mathsf{k}^2}\left\{\left[K^{(ij)} - \tau^{(ij)}\right] - h^{ij}\left(K - \tau\right)\right\}$$

$$= -\frac{\sqrt{h}}{2\mathsf{k}^2}\left[{}^{(0)}K^{(ij)} - h^{ij\,(0)}K\right] , \tag{4.9}$$

$$\pi_\phi \equiv \frac{\delta L}{\delta \dot{\phi}} = \frac{\sqrt{h}}{\mathcal{N}}\left(\dot{\phi} - \mathcal{N}^i \phi_{,i}\right) , \tag{4.10}$$

$$\pi^0 \equiv \frac{\delta L}{\delta \dot{\mathcal{N}}} = 0 , \tag{4.11}$$

$$\pi^i \equiv \frac{\delta L}{\delta \dot{\mathcal{N}}_i} = 0 . \tag{4.12}$$

Note 4.1 The reader is perhaps wondering why

$$-\frac{\sqrt{h}}{2\mathsf{k}^2}\left\{\left[K^{(ij)} - \tau^{(ij)}\right] - h^{ij}\left(K - \tau\right)\right\} = -\frac{\sqrt{h}}{2\mathsf{k}^2}\left[{}^{(0)}K^{(ij)} - h^{ij\,(0)}K\right]$$

was written out in full in the expression (4.9), instead of simply putting [26, 27]

$$-\frac{\sqrt{h}}{2\mathsf{k}^2}\left[{}^{(0)}K^{(ij)} - h^{ij\,(0)}K\right] ,$$

which is the pure gravity (general relativistic) case. The point is that when the action (4.8) is extended to obtain the N=1 SUGRA action, including the Rarita–Schwinger term, we will then have

$$-\frac{\sqrt{h}}{2\mathsf{k}^2}\left\{\left[K^{(ij)} - \tau^{(ij)}\right] - h^{ij}\left(K - \tau\right)\right\} = \mathcal{B}^{ij}(\psi_k^{[a]}) ,$$

with $\mathcal{B}^{ij}\left(\psi_k^{[a]}\right)$ an expression, i.e., a functional, that depends on the gravitino field $\psi_k^{[a]}$. But more on that later in this chapter. Let us just add that it really is needed when exhibiting the SUSY gauge structure of SUGRA in a Hamiltonian framework.

For the Hamiltonian,

$$H \equiv \int \mathrm{d}^3x \left(\pi^0 \dot{\mathcal{N}} + \pi^i \dot{\mathcal{N}}_i + \pi^{ij}\dot{h}_{ij} + \pi_\phi \dot{\phi}\right) - L$$

$$= \int \mathrm{d}^3x \left(\pi^0 \dot{\mathcal{N}} + \pi^i \dot{\mathcal{N}}_i + \mathcal{N}\mathcal{H}_\perp + \mathcal{N}_i \mathcal{H}^i\right) , \tag{4.13}$$

we have instead [34]

$$\mathcal{H}_{\perp} \equiv 2\mathsf{k}^2 \mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \frac{h^{1/2}\,{}^{(3)}R}{2\mathsf{k}^2} + \frac{h^{1/2}}{\mathsf{k}^2}\left[\tau^{(ij)}\tau_{(ij)} - \tau^2\right]$$
$$+\frac{1}{\mathsf{k}^2}\left(h^{1/2}\xi_{ij\perp}\xi^{ij}{}_{\perp} - 2h^{1/2}\tau_{[ij]}\xi^{ji}{}_{\perp} + h^{1/2}q^i\rho_i - 2h^{1/2}\rho^i_{\|i} + 2h^{1/2}\rho^i\rho_i\right)$$
$$+\frac{h^{1/2}}{2}\left(\frac{\pi_\phi^2}{h} + h^{ij}\phi_{,i}\phi_{,j} + 2V\right), \tag{4.14}$$

$$\mathcal{H}^i \equiv -2\pi^{ij}{}_{|j} + h^{ij}\phi_{,j}\pi_\phi , \tag{4.15}$$

where a double vertical bar $\|$ denotes covariant differentiation with respect to the spatial metric including the spin connection, and where

$$\mathcal{G}_{ijkl} = \frac{1}{2}h^{-1/2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right) \tag{4.16}$$

is the *DeWitt metric* once again.

4.1.2 Decomposition with Tetrad and Torsion

When dealing with a theory like SUGRA, where fermions and bosons are equally fundamental, it is essential to use the tetrad formalism rather than a metric representation [46–48]. The tetrad procedure involves introducing a pseudo-orthonormal basis $e^{\mu}{}_a$ of the tangential space at every point of the spacetime.[6]

In this section, we will use a 4-spinor representation. Besides being relevant again in Chap. 7 of Vol. II, it has the advantage of being more economical when presenting mathematical expressions. Only one spinor equation/label is needed,[7] the transition to Weyl spinors being straightforward. It is also easier when carrying out a $3+1$ spacetime decomposition for $N = 1$ SUGRA, because terms associated with torsion (i.e., with the presence of the gravitino) can be identified more quickly.

Let us therefore take the Lagrangian (of the first order formulation) for $N = 1$ SUGRA in (3.124), with the definition of terms as in, e.g., (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), and (3.136), where we recall something that will be important later, namely that the covariant derivative has *no* spacetime connection (i.e., Christoffel) term, this feature being a characteristic of

[6] Recall that a is the 'flat' index of the tetrad and runs from 0 to 3 (see also Appendix A). Indices $a, b, c, \ldots,$ are raised and lowered with η^{ab} and η_{ab}, respectively, where η^{ab} has the signature $(-,+,+,+)$. Spacetime indices μ, ν are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively. The relation between the spacetime metric and the tetrad elements can be written as $g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu$, $\eta^{ab} = g^{\mu\nu}e^a{}_\mu e^b{}_\nu$.

[7] Although in *some* cases for the 2-spinor notation, just one equation is sufficient for working purposes, the other just being the Hermitian conjugate.

$N = 1$ SUGRA. (This is called the *non-minimal* coupling in SUGRA.) For the $3+1$ spacetime decomposition of the $N = 1$ SUGRA action we will import the framework presented in the last section. The following references are well worth reading in this context [29, 30, 49, 34, 35, 40–44]. A selection of the relevant definitions and expressions can be found in Appendix A.

In essence, we initially proceed as in a theory with an Einstein–Hilbert action with torsion, i.e., a CSK setting (see Sect. 3.4.3), adding the Rarita–Schwinger action, where the corresponding $3 + 1$ decomposition also has to be implemented. Finally, substituting in the terms for torsion expressions dependent on the gravitino (a vector–spinor quantity), the Lagrangian for $N = 1$ SUGRA then[8] takes the (intermediate) form[9]

$$
\begin{aligned}
L = \frac{1}{2}\mathcal{N}h^{1/2}\Big\{ {}^{(3)}R + \left[K_{(ij)} - 2\tau_{(ij)}\right]K^{(ij)} - K(K - 2\tau) \\
- K_{[ij]}K^{[ij]} + 2\tau_{[ij]}K^{[ij]} - q^i\rho_i + 2\rho^i_{||i} - 2\rho^i\rho_i \Big\} \\
- \frac{\mathrm{i}}{2}\varepsilon^{\lambda\mu\nu\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho \ ,
\end{aligned}
\tag{4.17}
$$

where $\tau_{ij} \equiv 2\xi_{i\perp j}$, $\rho_i \equiv 2\xi_{ij}{}^j$, $q_i \equiv 4\xi_{i\perp}{}^\perp$ [see Appendix A for notation and definitions, and in particular (A.65), (A.87), (A.88), (A.89), (A.90), (A.91), and (A.92)]. In this setting the (intermediate) basic variables are the tetrad $e^a{}_\mu$, gravitino $\psi^{[a]}_\mu$, and torsion $\xi_{\mu\nu\lambda}$, whose canonical momenta are, respectively,

$$p^0{}_a = 0 \ , \tag{4.18}$$

$$
\begin{aligned}
p^i{}_a = -h^{1/2}\left[K^{(ik)} - \tau^{(ik)}e_{ka} - e^i{}_a(K - \tau)\right] \\
- \frac{\mathrm{i}}{4}h^{1/2}A^{\lambda\mu\perp\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu\left(\sigma^i{}_a - n_a\sigma^{\perp i}\right)\psi_\rho \\
- \frac{\mathrm{i}}{4}h^{1/2}A^{\lambda\mu i\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu\sigma_{\perp a}\psi_\rho + \frac{\mathrm{i}}{4}g^{1/2}A^{\lambda\mu k\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu e_{ka}\sigma^{\perp i}\psi_\rho \ ,
\end{aligned}
\tag{4.19}
$$

$$\pi^0 = 0 \ , \tag{4.20}$$

$$\pi^i = h^{1/2}\overline{\psi}_j\gamma^\perp\sigma^{ji} \ \longleftrightarrow \ \ell^i \equiv \pi^i - h^{1/2}\overline{\psi}_j\gamma^\perp\sigma^{ji} = 0 \ , \tag{4.21}$$

$$P^{\mu\nu\lambda} = 0 \ , \tag{4.22}$$

where $A^{\lambda\mu\nu\rho} \equiv \varepsilon^{\lambda\mu\nu\rho}/\mathcal{N}h^{1/2}$, $n_a = n^\mu e_{\mu a}$, and with the assistance of $\varepsilon^{ijk}\gamma_5\gamma_i = 2\mathrm{i}h^{1/2}\gamma^\perp\sigma^{jk}$, where $\sigma^{ij} \equiv \left(\gamma^i\gamma^j - \gamma^j\gamma^i\right)/4$.

---

[8] It is important to emphasize, if it is not already clear, that we formulate a CSK theory but with torsion *determined and explicitly written* in terms of a $\psi^{[a]}_\mu$ gravitino matter sector.

[9] To simplify, until further notice, we follow [34] and set $\mathsf{k}^2 = 1$.

We should make a few remarks in preparation for the rather long process of deriving the desired expressions:

- We define the quantity

$$\begin{aligned}\pi^{km} &\equiv \frac{1}{2}\left(p^{kd}e^{m}{}_{d} + p^{md}e^{k}{}_{d}\right) \\ &= -\sqrt{h}\left\{\left[K^{(km)} - \tau^{(km)}\right] - h^{km}\left(K - \tau\right)\right\} + \mathcal{B}^{km} ,\end{aligned} \tag{4.23}$$

where π^{ij} will act as canonical momenta to h_{ij}, and

$$\mathcal{B}^{km} \equiv \frac{\mathrm{i}}{4\mathcal{N}}\left[\varepsilon^{\lambda\mu k\rho}\overline{\psi}_{\lambda}\gamma_5\gamma_{\mu}\sigma^{\perp m}\psi_{\rho} - (k \leftrightarrow m)\right]$$

constitutes a rather important term. The tetrad $e^{a}{}_{\mu}$ and gravitino $\psi^{[a]}_{\mu}$ will be mixed here, as displayed within local SUSY. This is also a direct consequence of the non-minimal (derivative) coupling characterising $N = 1$ SUGRA.
- Equations (4.18), (4.20), and (4.21) are *first class* constraints, while (4.21) and (4.22) constitute *second class* constraints (see Appendix B).
- The usual Poisson brackets follow:

$$\left[e^{a}_{\mu}(x), p^{\nu}_{b}(x')\right]_{\mathrm{P}} = \delta^{\nu}_{\mu}\delta^{a}_{b}\delta(x, x') , \tag{4.24}$$

$$\left[\psi_{\mu[b]}(x), \pi^{\nu}_{[a]}(x')\right]_{\mathrm{P}} = \delta^{\nu}_{\mu}\delta_{[a][b]}\delta(x, x') = \left[\pi^{\nu}_{[a]}(x'), \psi_{\mu[b]}(x)\right]_{\mathrm{P}} , \tag{4.25}$$

$$\left[\xi_{\lambda\mu\nu}(x), P^{\rho\sigma\tau}(x')\right]_{\mathrm{P}} = \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\sigma}_{\mu}\delta^{\rho}_{\nu}\right)\delta^{\tau}_{\lambda}\delta(x, x') . \tag{4.26}$$

In order to deal with the second class constraints (4.21) [the constraint (4.22) will be dealt with below], indicating that fermions are self-conjugate in the action, we apply the method of Dirac brackets (see Exercise 4.1 and Appendix B). The Dirac brackets are then found in the form[10] (see (4.21))

$$\left[\psi_{m[a]}(x), \psi_{n[b]}(x')\right]_{\mathrm{D}} = \frac{1}{2}h^{-1/2}\left(\gamma_{\perp}\gamma_n\gamma_m\gamma_0\right)_{[a][b]}\delta(x, x') , \tag{4.27}$$

$$\begin{aligned}\left[p^{ia}(x), \psi_n(x')\right]_{\mathrm{D}} &= -\left[p^{ia}(x), \lambda^{m}\right]_{\mathrm{P}} D_{mn} \\ &= \frac{1}{2}\overline{\psi}_s\gamma^{a}\gamma_n\sigma^{si}\gamma_0\delta(x, x') ,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\left[p^{ia}(x), p^{jb}(x')\right]_{\mathrm{D}} &= -\left[p^{ia}(x), \lambda^{m}\right]_{\mathrm{P}} D_{mn}\left[\lambda^{n}, p^{jb}(x')\right]_{\mathrm{P}} \\ &= \frac{1}{2}h^{1/2}\overline{\psi}_s\gamma^{a}\sigma^{jr}\gamma_{\perp}\sigma^{is}\gamma^{b}\psi_r\delta(x, x') ,\end{aligned} \tag{4.29}$$

---

[10] It is worth pointing out the approach of [28, 34]. The authors use $h^{1/2}\psi_a$ instead of ψ_{μ}, as well as the *time gauge*, retrieving $\left[p^{ia}(x), h^{1/2}\psi_b(x')\right]_{\mathrm{D}} = \left[p^{ia}(x), p^{jb}(x')\right]_{\mathrm{D}} = 0$. What is the time gauge? In brief, it consists in removing three of the six $SO(3, 1)$ degrees of freedom, leaving only the $SO(3)$ spatial rotations.

where

$$\left[p^{ia}(x), \lambda^m\right]_{\mathrm{P}} = -\frac{\mathrm{i}}{2}\varepsilon^{ism}\overline{\psi}_s\gamma_5\gamma^a \ ,$$

π^k has been replaced by $h^{1/2}\overline{\psi}_j\gamma^{\perp}\sigma^{jk}$, and

$$D_{ij[a][b]} \equiv -\frac{1}{2\sqrt{h}}\left(\gamma_{\perp}\gamma_j\gamma_i\gamma_0\right)_{[a][b]} \ . \tag{4.30}$$

We can now consider the Hamiltonian

$$H = \dot{e}_{ka}p^{ka} + \dot{\psi}_{\rho}\pi^{\rho} - L + \left(\dot{e}_{0a}p^{0a} + \dot{\xi}_{\mu\nu\lambda}P^{\mu\nu\lambda}\right) \ , \tag{4.31}$$

where $\left(\dot{e}_{0a}p^{0a} + \dot{\xi}_{\mu\nu\lambda}P^{\mu\nu\lambda}\right)$ is zero. Here is an itemized set of *ten* steps (for more details, see [34]):

1. The computation of the term $\dot{e}_{ka}p^{ka}$ can be expressed in terms of $\dot{h}_{ik}$ (retrieved from the Einstein–Hilbert sector), which will be useful (a) when using π^{km} above, and (b) to connect with the general relativistic setting (see Chap. 2).
2. Hence, time derivatives, in particular those involving $\dot{h}_{ik}$, are then eliminated using (4.23) for π^{km}, together with the explicit expression for the extrinsic curvature K. The reader, if attempting to reproduce these lengthy calculations, should be prepared to use $\mathcal{B}^{km}$ soon.
3. We recall some intermediate definitions:

   - Spacetime derivatives with respect to the full connection, i.e., including Christoffel terms and spin connections, are denoted by $\wr$.
   - 3-surface derivatives with the full connections, i.e., including Christoffel terms and spin connections, are denoted by $\|$.
   - 3-surface derivatives including only Christoffel terms are denoted by $|$.
   - We will also use the alternative notation (see Sect. 3.4)

$$D_{\nu} \equiv \partial_{\nu} + \frac{1}{2}\omega_{\nu ab}[e^c_{\mu}, \psi^{[a]}_{\mu}]\sigma^{ab} \ , \tag{4.32}$$

$$\omega_{\nu ab}[e^c_{\mu}, \psi^A_{\mu}] \equiv \omega_{\nu ab}(e^c_{\mu}) + \kappa_{\nu\mu\lambda}e^{\mu}_a e^{\lambda}_b \ , \tag{4.33}$$

$$^{(3)}D_k \equiv \nabla_k = \partial_k + \frac{1}{2}\omega_{kab}[e^c_{\mu}, \psi^A_{\mu}]\sigma^{ab} \ , \tag{4.34}$$

$$^{(3s)}D_k \equiv \tilde{\nabla}_k \equiv \partial_k + \frac{1}{2}\omega_{kab}[e^c_{\mu}]\sigma^{ab} \ . \tag{4.35}$$

4. The Hamiltonian would have the form $H = \mathcal{N}(\cdots) + \mathcal{N}^i(\ldots) + \cdots$, i.e., the structure of the momentum and what will become SUSY extended Wheeler–DeWitt constraints are there, *but* the generator of (local) SUSY transformations is yet to be extracted. This will proceed from the term with $\mathcal{B}_{ij}\pi^{ij}$, bearing in mind (4.23), and $\varepsilon^{\lambda\mu k\rho}\overline{\psi}_{\lambda}\gamma_5\gamma_{\mu}\nabla_k\psi_{\rho}$.

5. But before that, one further step is relevant. If the constraints present in the Hamiltonian are to be *first class* [50, 51, 36, 52], and therefore associated with generators of gauge invariances [37] (see Appendix B), *all* terms in H that are quadratic in ψ_0, $\mathcal{N}$, and $\mathcal{N}^i$, which are identified as Lagrange multipliers, must either vanish or cancel.[11] The fact is that *they do cancel*, in particular from the analysis of $\mathcal{B}^2$ and τ^2, therefore securing the property of (local) SUSY invariance for the theory of $N = 1$ SUGRA in the Hamiltonian setting.
6. The reader will have noticed by now the presence of terms in the contorsion $\kappa_{\nu\mu\lambda} e^{\mu}{}_a e^{\lambda}{}_b$. But requiring $P^{\mu\nu\lambda} = 0$ to be *conserved*, it follows that

$$\xi_{\mu\nu\lambda} = -\frac{1}{4}\overline{\psi}_\mu \gamma_\lambda \psi_\nu \ . \tag{4.36}$$

7. Dealing with the constraints $P^{\mu\nu\lambda} = 0$ and $\xi_{\mu\nu\lambda} = -\overline{\psi}_\mu \gamma_\lambda \psi_\nu / 4$ through *extended* Dirac brackets allows one to use $\xi_{\mu\nu\lambda}$ or $-\overline{\psi}_\mu \gamma_\lambda \psi_\nu / 4$ in subsequent expressions.
8. Focus now on the terms extracted from the Rarita–Schwinger action for the gravitino, e.g., $\varepsilon^{\lambda\mu\hat{k}\rho}\overline{\psi}_\lambda \gamma_5 \gamma_\mu \tilde{\nabla}_k \psi_\rho$ ($V^{\hat{k}} \equiv V^k - \mathcal{N}^k V^{\perp}/\mathcal{N}$, see Appendix A). An expression of the form $\mathcal{N}(\ldots) + \mathcal{N}^i(\ldots) + \overline{\psi}_0(\ldots)$ is now established, and the SUSY generators $\mathcal{S}$ thus emerge.
9. Moreover, from $\varepsilon^{\lambda\mu\nu\rho}\overline{\psi}_\lambda \gamma_5 \gamma_\mu \kappa_{\nu\gamma\sigma} \sigma^{\gamma\sigma} \psi_\rho$ in the Rarita–Schwinger action and

$$\tau_{(ij)}\tau^{(ij)} - \tau^2 + \frac{\mathcal{B}^{ij}\mathcal{B}_{ij} - \mathcal{B}^2/2}{g}$$

   in the Einstein–Hilbert sector, and with the assistance of (4.36), we retrieve the expected cancelling of terms quadratic in ψ_0, $\mathcal{N}$, and $\mathcal{N}^i$. The methodical explorer should ponder here on the fact that this is *only possible* due to the non-minimal (derivative) coupling,[12] characteristic of $N = 1$ SUGRA.
10. The form of the constraints $\mathcal{H}_\perp$ and $\mathcal{H}_i$ in $N = 1$ SUGRA is basically achieved, and with some rearrangements, the full SUSY constraints appear. The total Hamiltonian is then[13]

---

[11] This means that the constraints we will use ($\mathcal{H}_\perp, \mathcal{H}_i, \mathcal{S}, \mathcal{J}^{ab}$) contain only the canonical variables and *no* Lagrange multipliers. This secures a simpler Hamiltonian, and more importantly, it means that the constraints will satisfy $\left[C_m, C_n\right]_{\mathrm{D}} = \rho^k_{mn} C_k$, which is essential when applying the *Teitelboim procedure* [34, 35, 40, 41].

[12] If a minimally coupled action, with $\wr$ instead of D_ν, is employed, then SUSY is lost, therefore confirming what was indicated in Chap. 3 concerning the form of the $N = 1$ SUGRA action.

[13] Remember, the derivative ∇_j contains torsion in the form of terms quadratic in the gravitino.

$$H = \mathcal{N}\mathcal{H}'_\perp + \mathcal{N}^i\mathcal{H}'_i + \overline{\psi}_0\mathcal{S}' \ , \tag{4.37}$$

$$\mathcal{H}'_\perp = \frac{1}{2}\Bigg[h^{-1/2}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - h^{1/2\ (3)}R - \mathrm{i}\varepsilon^{ijk}\overline{\psi}_i\gamma_5\gamma_\perp\nabla_j\psi_k - \frac{1}{16}h^{1/2}\overline{\psi}_i\gamma_\perp\psi_j\overline{\psi}^i\gamma_\perp\psi^j\Bigg] , \tag{4.38}$$

$$\mathcal{H}'_m = -h_{mi}\pi^{ik}{}_{|k} + \pi^k\left(\nabla_k\psi_m - \nabla_m\psi_k\right) + \frac{1}{2}\overline{\psi}_m\gamma_j\psi_i\pi^{ij} - h^{1/2}\overline{\psi}_m\gamma_\perp\sigma^{jk}\nabla_j\psi_k + \frac{1}{4}h^{1/2}h_{mi}\left(\overline{\psi}^k\gamma_\perp\psi^i\right)_{||k} + \frac{1}{8}h^{1/2}\overline{\psi}_m\gamma_\perp\psi^s\overline{\psi}_s\gamma^k\psi_k \tag{4.39}$$

$$\mathcal{S}' = \mathrm{i}\varepsilon^{ijk}\gamma_5\gamma_i\nabla_j\psi_k - \frac{1}{2}\gamma_j\psi_i\pi^{ij} - \frac{1}{8}h^{1/2}\gamma_i\psi_j\overline{\psi}^i\gamma_\perp\psi^j \ . \tag{4.40}$$

Note 4.2 Concerning the structure of the above Hamiltonian (4.37), (4.38), (4.39), and (4.40), the following comments should be made:

- The first three terms in (4.38) represent the contribution from the gravitational field in the Einstein–Hilbert sector and a curved spacetime setting with the Rarita–Schwinger (gravitino) field.
- The first term in (4.40) is the curved spacetime translation of the SUSY generator, but it is the next term $\gamma_j\psi_i\pi^{ij}$ that induces

$$\left[h_{ij}, \mathcal{S}'\right]_{\mathrm{D}} \sim \gamma_i\psi_j + \gamma_j\psi_i \ , \tag{4.41}$$

$$\left[e^a_k, \gamma_i\psi_j\pi^{ij}\right]_{\mathrm{D}} \sim \psi_k \ , \tag{4.42}$$

$$\left[\psi_k, \gamma_i\psi_j\pi^{ij}\right]_{\mathrm{D}} \sim \pi^{kl} \ , \tag{4.43}$$

  therefore mixing the bosonic and fermionic content,[14] i.e., part of the SUSY properties of $N = 1$ SUGRA.
- Within $\mathcal{H}'_i$, we find terms for generating spatial coordinate transformations of the gravitational variables, while $\pi^k\left(\nabla_k\psi_m - \nabla_m\psi_k\right)$ is associated with gauge fields with vectorial structure, and $\overline{\psi}_m\gamma_j\psi_i\pi^{ij}$ is related to the derivative coupling (note that $\gamma_j\psi_i\pi^{ij}$ is present in $\mathcal{S}'$).

[14] As remarked in [34], this mechanism is specific to SUGRA, whereby the gauge symmetry is neither purely external or internal, and caused by the *specific* derivative coupling when taking SUSY into a *curved* space.

- The other terms have no simple interpretation, but are crucial for closing the algebra, i.e., for the use of Dirac brackets among the generators to be self-consistent.

4.1.3 Algebra of Constraints

Two questions come to mind at this point. First, where are the constraints in (4.37), (4.38), (4.39), and (4.40)? This is simple. When we require the constraints (4.18) and (4.20) to be conserved in time, we get

$$\pi^0 = 0 \implies \mathcal{S}' = 0 \,, \tag{4.44}$$

$$p^0{}_a = 0 \implies \begin{cases} \mathcal{H}'_\perp = 0 \,, \\ \mathcal{H}'_i = 0 \,, \end{cases} \tag{4.45}$$

i.e., the SUSY constraints $\mathcal{S}' = 0$ and the momentum and Hamiltonian constraints, $\mathcal{H}'_i = 0$ and $\mathcal{H}'_\perp = 0$, respectively, satisfying

$$\left[\pi^0, \mathcal{S}'\right]_\mathrm{D} = \left[p^0{}_a, S'\right]_\mathrm{D} = 0 \,, \tag{4.46}$$

$$\left[\pi^0, \mathcal{H}'_\perp\right]_\mathrm{D} = \left[p^0_a, \mathcal{H}'_\perp\right]_\mathrm{D} = 0 \,, \tag{4.47}$$

$$\left[\pi^0, \mathcal{H}'_i\right]_\mathrm{D} = \left[p^0{}_a, \mathcal{H}'_i\right]_\mathrm{D} = 0 \,, \tag{4.48}$$

as required for first class constraints (from (4.44) and (4.45), they are also secondary, see Appendix B). But it remains to compute the Dirac brackets among the constraints to confirm their status as first class.

And it is here that we come to the second question: Where is the Lorentz constraint associated with local Lorentz invariance for the spinors? In fact, it is from discussing it at this stage that we can also justify the presence of the primes in the constraints (4.44) and (4.45) (see Note 2.5, Exercise 2.1, and [48]). From (4.19) [46, 47, 53, 54, 32, 33], we now find that

$$\mathcal{J}^{ab} \equiv p^{ka} e^b{}_k - p^{kb} e^a{}_k + \pi^k \sigma^{ab} \psi_k = 0 \,, \tag{4.49}$$

from which it follows that

$$\left[e_{ka}, \int \mathrm{d}x' \mathcal{M}_{ab}(x') \mathcal{J}^{ab}\right]_\mathrm{D} = 2\mathcal{M}_{ab} e^b_k \,,$$

therefore generating local Lorentz transformations. $\mathcal{J}^{ab}$ is thus the (primary) Lorentz constraint.

To check the closure of the SUGRA algebra, we must therefore compute the Dirac brackets of (4.44), (4.45), and (4.49), using of course (4.27), (4.28), and (4.29). A simplified and extremely elegant approach, fully equivalent and much more economical, is to employ the *Teitelboim procedure*. Briefly, the strategy is as follows.

To establish a relation $[C_m, C_n]_{\mathrm{D}} = \rho^k_{mn} C_k$ for the first class constraints C_n, we do *not* need to actually compute $[C_m, C_n]_{\mathrm{D}}$. We can retrieve ρ^k_{mn} by an ingenious approach.[15] Let us see how. In order for the $N = 1$ SUGRA action, in the Hamiltonian representation, to be invariant under the transformations

$$p \longrightarrow p + \left[p, C_m\right]_{\mathrm{D}} \varepsilon^m \ , \qquad q \longrightarrow q + \left[q, C_m\right]_{\mathrm{D}} \varepsilon^m \ ,$$

for the canonical momenta and coordinates, respectively, the Lagrange multipliers associated with translations, rotations, and SUSY, respectively, are taken to be the covariant time component of the gauge fields, e.g., $\ell \equiv \{e^a_0, \omega_{0ab}, \psi_0\}$, and should transform according to

$$\ell^m \longrightarrow \ell^m + \dot{\varepsilon}^m + \rho^k_{mn} \varepsilon^m \ell^n \ .$$

Therefore, to retrieve ρ^k_{mn}, we determine how the Lagrange multipliers change! A significant (and non-obvious) ingredient remains to be introduced. For translations of e^a_0, ω_{0ab}, and ψ_0 to be in the required form, a modification is necessary. It includes *suitably chosen* rotations and SUSY transformations, leaving the action invariant. From δe^a_0, $\delta\omega_{0ab}$, and $\delta\psi_0$, we proceed to establish (4.51)–(4.65) below, of course assuming that the constraints have *no* quadratic terms in the Lagrange multipliers:

- Modify the form of $\mathcal{H}'_\perp$, $\mathcal{H}'^i$, and $\mathcal{S}'$ by including terms proportional to $\mathcal{J}^{ab}$.
- The Hamiltonian is then rewritten as[16]

$$H = \mathcal{N}\mathcal{H}_\perp + \mathcal{N}^i\mathcal{H}_i + \overline{\psi}_0\mathcal{S} - \frac{1}{2}\omega_{0ab}\mathcal{J}^{ab} \ . \tag{4.50}$$

- Notice the use of $-\omega_{0ab}/2$ as Lagrange multiplier, conveying the purpose of having a term with a geometrical interpretation: ω_{0ab} is the complete spin connection including torsion.
- In full, we now write [34]:

---

[15] The ρ^k_{mn} are the structure constants of the algebra, which can be functionals of $q \equiv \{e, \psi\}$ and the corresponding canonically conjugate momenta p.

[16] The Hamiltonian H in (4.37) does not contain arbitrary rotations in a self-evident manner, so the presence of $\mathcal{J}^{ab}$ in the Teitelboim procedure cures this aspect with an appropriate geometrical structure.

$$
\begin{aligned}
\mathcal{H}_{\perp} &= \mathcal{H}'_{\perp} + \frac{1}{2}\left(2\mathcal{J}^{\perp k}{}_{,k} - \frac{1}{4}\overline{\psi}_i\gamma_{\perp}\psi_j\mathcal{J}^{ij}\right) \\
&= \frac{1}{2}\Bigg[h^{-1/2}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - h^{1/2\,(3)}R - \mathrm{i}\varepsilon^{ijk}\overline{\psi}_i\gamma_5\gamma_{\perp}\nabla_j\psi_k \\
&\quad -h^{1/2}\left(\overline{\psi}^i\gamma^j\psi_j\right)_{||i} + \frac{1}{16}h^{1/2}\overline{\psi}^s\gamma_{\perp}\psi^t\overline{\psi}_s\gamma_{\perp}\psi_t + 2p^{k\perp}{}_{,k} \\
&\quad +\frac{1}{2}\overline{\psi}_i\gamma_{\perp}\psi_j p^{ij}\Bigg],
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
\mathcal{H}_m &= \mathcal{H}'_m + \frac{1}{2}\Bigg[h_{mj}\mathcal{J}^{kj}{}_{,k} + \frac{1}{2}\overline{\psi}_m\gamma_{\perp}\mathcal{J}^{m\perp} - \frac{1}{4}\overline{\psi}_i\gamma_m\psi_j\mathcal{J}^{ij} \\
&\qquad - \frac{2}{h^{1/2}}\left(\pi_{km} - \frac{1}{2}\pi h_{km}\right)\mathcal{J}^{\perp m}\Bigg] \\
&= -h_{mi}\pi^{ik}{}_{|k} + g_{mj}\partial_k p^{[kj]} + \pi^k\left(\nabla_k\psi_m - \nabla_m\psi_k\right) + \frac{1}{2}\overline{\psi}_m\gamma_j\psi_i\pi^{ij} \\
&\quad -h^{1/2}\overline{\psi}_m\gamma_{\perp}\sigma^{jk}\nabla_j\psi_k + \frac{1}{4}\overline{\psi}_m\gamma_{\perp}\psi_i p^{i\perp} - \frac{1}{8}h^{1/2}\overline{\psi}_m\gamma_{\perp}\psi_j\overline{\psi}^j\gamma^k\psi_k \\
&\quad +\frac{1}{4}\overline{\psi}_i\gamma_m\psi_j\mathcal{J}^{ij} - \frac{2}{h^{1/2}}\left(\pi_{km} - \frac{1}{2}\pi h_{km}\right)\mathcal{J}^{\perp m}\ ,
\end{aligned}
\tag{4.52}
$$

$$
\begin{aligned}
\mathcal{S} &= \mathcal{S}' + \frac{1}{4}\overline{\psi}_j\gamma_i\mathcal{J}^{ij} - \frac{1}{2}\gamma_{\perp}\psi_k\mathcal{J}^{\perp k} \\
&= 2h^{1/2}\gamma_{\perp}\sigma^{jk}\nabla_j\psi_k - \frac{1}{2}p^i_b\gamma^b\psi_i + \frac{1}{4}h^{1/2}\gamma_{\perp}\psi_i\overline{\psi}^i\gamma^j\psi_j\ .
\end{aligned}
\tag{4.53}
$$

with

$$
p^{ij} = p^{ia}e^j_a\ , \qquad p^{k\perp} \equiv -n^a p^k_a\ . \tag{4.54}
$$

It is from the Hamiltonian (4.50) above that the Teitelboim process allows one to compute the Dirac brackets among the generators. Moreover, it is worth noting that it is from (4.53) that the SUSY transformation $\delta e^a_k \sim \gamma^a\psi_k$ is retrieved.

- In more detail, the Dirac bracket algebra is thus:

$$
\left[\mathcal{S}(x), \overline{\mathcal{S}}(x')\right]_{\mathrm{D}} = \frac{1}{2}\gamma^a\mathcal{H}_a\delta(x,x')\ , \tag{4.55}
$$

$$
\left[\mathcal{S}(x), \mathcal{H}_c(x')\right]_{\mathrm{D}} = \frac{1}{2}\Upsilon_{cab}\mathcal{J}^{ab}\delta(x,x'), \tag{4.56}
$$

$$
\left[\mathcal{S}(x), \mathcal{J}^{ab}(x')\right]_{\mathrm{D}} = \sigma^{ab}\mathcal{S}\delta(x,x')\ , \tag{4.57}
$$

$$
\left[\mathcal{H}_a(x), \mathcal{H}_b(x')\right] = \left(\frac{1}{2}\Theta_{abcd}\mathcal{J}^{cd} + \overline{\vartheta}_{ab}\mathcal{S}\right)\delta(x,x')\ , \tag{4.58}
$$

$$\left[\mathcal{H}_c(x), \mathcal{J}^{ab}(x')\right] = \left(\delta^a_c \mathcal{H}^b - \delta^b_c \mathcal{H}^a\right) \delta(x, x') \,, \tag{4.59}$$

$$\left[\mathcal{J}^{ab}(x), \mathcal{J}^{cd}(x')\right] = \left(\eta^{ac} \mathcal{J}^{bd} - \eta^{ad} \mathcal{J}^{bc} - \eta^{bc} \mathcal{J}^{ad} - \eta^{bd} \mathcal{J}^{ac}\right) \delta(x, x') \,, \tag{4.60}$$

where[17]

$$\mathcal{H}_a \equiv \mathcal{H}_\mu e^\mu_a = -\mathcal{H}_\perp n_a + \mathcal{H}_k e^k_a \,, \tag{4.61}$$

$$\Upsilon_{\mu ab} \equiv \mathrm{i}\gamma_5 \left(\gamma_\mu \vartheta^*_{ab} + \frac{1}{2} e_{\mu a} \vartheta^*_{bd} \gamma^d - \frac{1}{2} e_{\mu b} \vartheta^*_{ad} \gamma^d\right) \,, \tag{4.62}$$

$$\vartheta_{\mu\nu} \equiv D_\mu \psi_\nu - D_\nu \psi_\mu \,, \tag{4.63}$$

$$\vartheta^{*\mu\nu} \equiv \frac{1}{2}\left[-g^{(4)}\right]^{-1/2} \varepsilon^{\mu\nu\sigma\tau} \vartheta_{\sigma\tau} \,, \tag{4.64}$$

$$\Theta_{\nu\mu cd} \equiv R_{\nu\mu cd} - \overline{\psi}_\nu \Upsilon_{\mu cd} + \overline{\psi}_\mu \Upsilon_{\nu cd} \,. \tag{4.65}$$

Note 4.3 Perhaps the most relevant feature is (4.55), since it points to the *square-root* nature of SUGRA, concerning general relativity. To be more precise, similarly to the relations between Dirac and Klein–Gordon equations, the SUSY constraints act as a square-root (see Exercise 4.2) for the generator of general coordinate transformations, extending the global SUSY property $\left\{Q, \overline{Q}\right\} \sim \gamma^\mu \mathcal{P}_\mu$ to the wider landscape of $N = 1$ SUGRA. [Physical states in the quantum theory must satisfy $\mathcal{S}\,|\Psi\rangle = 0$, but this implies $\left(\mathcal{S}\overline{\mathcal{S}} + \overline{\mathcal{S}}\mathcal{S}\right)|\Psi\rangle = 0 \Longrightarrow \mathcal{H}_a\,|\Psi\rangle = 0$.] SUSY generators are thus *more fundamental* than those in (4.61).

Note 4.4 Before proceeding with the next section in which we present and justify the 2-spinor transcription of the above, let us add a few comments, for these constitute both standpoints and horizons for landscapes to be explored. The Hamiltonian formulation allows one to consider a geometrical formulation of $N = 1$ SUGRA, where the graviton and gravitino constitute different components of the same field. Related by gauge transformations, they have equivalent roles within a single entity, and this relationship goes well beyond a simple coupling of fields.

[17] $\mathcal{H}_a = \mathcal{H}_\mu e^\mu{}_a$ are the generators of *localized* translations, being the projection of the surface deformations $\mathcal{H}_\mu$ along the tetrad directions $e^\mu{}_a$.

Let us be more precise:

- When addressing the Hamiltonian formulation of general relativity in Chap. 2, we mentioned that a Riemannian structure is extracted, from the corresponding constraints (generators) $\mathcal{H}_\perp$ and $\mathcal{H}_i$ satisfying the relations (2.74), (2.75), and (2.76). What about $N = 1$ SUGRA? That is, from (4.51)–(4.65), what geometry is retrieved? What deformations of which hypersurfaces are they related to, and in what type of space? These questions merit investigation, and the determined fellow explorer is invited to contribute. But be warned: it is a complex issue.
- Just recall that in 4D spacetime, the geometry can be whatever we feed in, *unless* (2.74), (2.75), and (2.76) *forces it* to be Riemannian. In $N = 1$ SUGRA we have the generators for deformations in (four) ordinary (bosonic) spacetime coordinates and (four) additional (fermionic) Grassmanian coordinates, associated with $\mathcal{H}_\perp$, $\mathcal{H}_i$, and $\mathcal{S}$.
- These thus imply an 8D space (a *super*Riemannian space) in which those deformations take place. In particular, the presence of torsion *may mean* that we are using a non-coordinate basis for this geometry [34]. But additional discussions can be found in Chap. 4 of Vol. II.

4.1.4 Two-Component Spinor Representation

In the previous sections, we mainly followed the framework introduced by M. Pilati [34], also discussed by Teitelboim [40, 41] and others [32, 33, 35]. However, a 2-component spinor notation also reveals important canonical features, in particular, when discussing the subsequent canonical quantization of $N = 1$ SUGRA [25, 27, 55]. For this reason, we now briefly discuss the Hamiltonian formulation of $N = 1$ SUGRA. For details on the conversion between 4-component and 2-component spinors, see Appendix A, Exercise 4.3, and [56–58, 13].

In the action of $N = 1$ SUGRA in 4D spacetime [12], a 2-component spinor structure [25, 27] gives (see Sect. 3.4)

$$S[e, \psi] = \int \mathrm{d}^4x \left[\frac{1}{2\mathsf{k}^2} \det(e_\mu{}^a) \mathcal{R} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \left(\overline{\psi}{}^{A'}_\mu e_{AA'\nu} D_\rho \psi^A_\sigma + D_\rho \overline{\psi}{}^{A'}_\sigma e_{AA'\nu} \psi^A_\mu \right) \right] . \quad (4.66)$$

Here and henceforth, we adopt the notation and definitions introduced in [59], according to which, if the fields $e_i^{AA'}$ and ψ_i^A appear in the argument of a functional, the indices are often omitted for simplicity. For example, we write $S[e, \psi]$ instead of $S[e_i^{AA'}, \psi_i^A]$.

Note 4.5 The following points should be noted:

- Consider the decomposition of the Rarita–Schwinger sector:

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu^{A'} e_{AA'\nu} D_\rho \psi_\sigma^A + D_\rho \overline{\psi}_\sigma^{A'} e_{AA'\nu} \psi_\mu^A\right) ,$$

  in comparison with the term used in Sect. 3.4 for the gravitino field ψ_μ^A.
- The covariant derivative D_ρ acts only on the spinor indices and is defined with the help of the spin connection forms $\omega^A{}_{B\rho}$ and $\overline{\omega}^{A'}{}_{B'\rho}$. [See, e.g., Appendix A and Sect. A.4 for more details. This is essentially (3.127) in 2-spinor notation.]
- The reader should also remember that the Ricci curvature scalar in (4.66) is now a function of both the tetrad $e_{AA'\nu}$ and the gravitino ψ_μ^A, which can also be represented in terms of torsion and contorsion terms. This does imply significant physical changes in contrast with purely bosonic quantum cosmology and quantum gravity. This important feature will become evident throughout Chap. 5, for example.

Following the approach implemented in Sect. 4.1.2, it is a straightforward matter[18] to rewrite the spinorial setting for the canonical fields, in terms of the spatial components of the tetrad $e_i^{AA'}$ and the gravitino ψ_i^A and $\overline{\psi}_i^{A'}$ [25–27, 60, 34]. The momentum conjugate to the spatial tetrad is then given by

$$p^i_{AA'} = \frac{\delta S}{\delta \dot{e}_i^{AA'}} , \tag{4.67}$$

where the dot represents the partial derivative with respect to the timelike direction,[19] and the conjugate momenta for the gravitino are[20]

[18] $\mathcal{N}$ and $\mathcal{N}^i$ of (A.47) are Lagrange multipliers, as are ψ_0^A and $\overline{\psi}_0^{A'}$.

[19] Note the equivalent representation in terms of a symmetrized version (see Appendix A and (4.19), (4.20), (4.21), (4.22), and (4.23)):

$$\pi^{ij} \equiv -\frac{1}{2}p^{(ij)} , \qquad p^{ij} = -e^{AA'j} p^i_{AA'} . \tag{4.68}$$

[20] The momentum conjugate to $\overline{\psi}_i^{A'}$ is represented by $\tilde{\pi}^i_{A'}$, as it is *minus* the Hermitian conjugate of π^i_A [26, 27].

$$\pi^i_A = \frac{\delta S}{\delta \dot{\psi}^A_i} = -\frac{1}{2}\varepsilon^{ijk}\overline{\psi}^{A'}{}_b e_{AA'k} ,$$
$$\tilde{\pi}^i_{A'} = \frac{\delta S}{\delta \dot{\overline{\psi}}^{A'}_i} = \frac{1}{2}\varepsilon^{ijk}\psi^A{}_b e_{AA'k} . \tag{4.69}$$

From the above, the Hamiltonian is [25–27]

$$H = \int \mathrm{d}^3x \Big[\mathcal{N}({}_1\mathcal{H}_\perp) + \mathcal{N}^i({}_1\mathcal{H}_i) + \psi^A_0({}_1\mathcal{S}_A) + {}_1\overline{\mathcal{S}}_{A'}\overline{\psi}^{A'}_0$$
$$+\mathcal{M}_{AB}\mathcal{J}^{AB} + \overline{\mathcal{M}}_{A'B'}\overline{\mathcal{J}}^{A'B'}\Big] , \tag{4.70}$$

with the following remarks:

- **Hamiltonian and Diffeomorphism Constraints.** In the absence of matter fields, the Hamiltonian constraint[21] $\mathcal{H}_\perp \cong 0$ and the diffeomorphism (or momentum) constraints $\mathcal{H}_i \cong 0$, take the explicit form[22]

$$ {}_1\mathcal{H}_\perp = 2\mathsf{k}^2\,\mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}\,{}^{(3)}R}{2\mathsf{k}^2} + \frac{\mathsf{k}^2}{8}\sqrt{h}\,n_{AA'}\overline{\psi}^{A'}_{[i}\,\psi^A_{j]}n^{BB'}\overline{\psi}^{[i}_{B'}\psi^{j]}_B$$
$$+\frac{1}{2}\varepsilon^{ijk}\overline{\psi}^{A'}_i n_{AA'}\,{}^{(3)}D_j\psi^A_k + \frac{1}{2}\varepsilon^{ijk}({}^{(3)}D_j\overline{\psi}^{A'}_k)n_{AA'}\psi^A_i \tag{4.71}$$

and

$${}_1\mathcal{H}_i = -2h_{ij}\,{}^{(3s)}\nabla_k\pi^{jk} + \frac{\mathrm{i}}{2}\sqrt{h}\,h_{ij}\,{}^{(3)}\nabla_k\left(n^{AA'}\overline{\psi}^{[j}_{A'}\psi^{k]}_A\right)$$
$$+\frac{\mathsf{k}^2}{2}\sqrt{h}\,n_{AA'}\overline{\psi}^{A'}_{[j}\,\psi^A_{i]}e^{BB'}_k\overline{\psi}^{[j}_{B'}\psi^{k]}_B - 2\mathsf{k}^2\mathrm{i}\pi^{jk}e_{AA'j}\overline{\psi}^{A'}_{[i}\,\psi^A_{k]}$$
$$+\frac{1}{2}\varepsilon^{jkl}\left[\overline{\psi}^{A'}_j e_{AA'i}\,{}^{(3)}D_k\psi^A_l + {}^{(3)}D_k(\overline{\psi}^{A'}_l)e_{AA'i}\psi^A_j\right] . \tag{4.72}$$

The suffix 1 distinguishes this from another form to be introduced in Sect. 4.2, corresponding to a different combination of these constraints. The expressions should be compared with (4.37), (4.38), (4.39), and (4.40) or (4.51), (4.52), and (4.53).

The fermionic terms, i.e., the last three terms of (4.71) and all but the first term of (4.72), are of paramount importance in explaining the novel features of SUGRA with regard to gravitational dynamics, in particular the semiclassical limit (see Chap. 4 of Vol. II), contrasting it with the purely general relativistic case (see

---

[21] See Appendix B. The symbol $\cong$ for *weak* equality is employed here.

[22] Note that ${}^{(3)}R$ is the 3D scalar curvature (A.96). Moreover, ${}^{(3s)}\nabla_k$ and ${}^{(3)}\nabla_k$ correspond to the usual spatial covariant derivative without torsion and with torsion, respectively (see Appendix A).

Chap. 2 of Vol. I). In fact, it would be from the analysis of such terms that one could investigate and eventually probe *if* and *how* an imprint of an SQC stage might be present in the observable universe (see Chap. 4 of Vol. II).

- **Lorentz Constraints.** The explicit form of the Lorentz constraints (see also (4.49)) is given by

$$ {}_1\mathcal{J}_{AB} = e_{(A}{}^{A'i} p_{B)A'i} + \psi^i_{(A}\pi_{B)i} \cong 0 \,, \tag{4.73} $$

where ${}_1\overline{\mathcal{J}}_{AB}$ is its Hermitian conjugate.

- **Supersymmetry Constraints.** One of the SUSY constraints reads

$$ \begin{aligned} {}_1\mathcal{S}_A &= \varepsilon^{ijk} e_{AA'i}{}^{(3)}D_j \overline{\psi}^{A'}_k + \mathsf{k}^2 \mathrm{i}\pi^{ij} e_{AA'_i} \overline{\psi}^{A'}_j \\ &+ \frac{\mathsf{k}^2}{4}\left(\sqrt{h} e_{AA'i} \overline{\psi}^{A'}_j n^{BB'} \overline{\psi}^{[i}_{B'} \psi^{j]}_B - \mathrm{i}\varepsilon^{ijk} n_{AA'} \overline{\psi}^{A'}_j n_{BB'} \overline{\psi}^{B'}_k \psi^B_i \right) \cong 0 \,, \end{aligned} \tag{4.74} $$

where we recall ${}^{(3)}D_j$ is the covariant derivative acting on the spinor indices (see Appendix A), with $\overline{\mathcal{S}}_{A'}$ obtained as the Hermitian conjugate of $\mathcal{S}_A$ (see (4.51), (4.52), and (4.53)).

Similarly to what was done in Sect. 4.1.3 [26], we can rewrite the Hamiltonian with *new* Lagrange multipliers for the Lorentz constraint. Using ω_{AB0}, $\overline{\omega}_{A'B'0'}$, we subsequently write *new* expressions (labeled by $1'$), e.g.,

$$ \begin{aligned} {}_{1'}\mathcal{S}_A &= \varepsilon^{ijk} e_{AA'i}{}^{(3)}D_j \overline{\psi}^{A'}_k - \frac{\mathrm{i}}{2} p^i_{AA'} \overline{\psi}^{A'}_i \\ &+ \frac{\mathsf{k}^2}{4}\left(2\sqrt{h} n_{AA} \overline{\psi}^{A'}_i \overline{\psi}^{[j}_{B'} \psi^{i]}_B - \mathrm{i}\varepsilon^{ijk} n_{AA'} \overline{\psi}^{A'}_j n_{BB'} \overline{\psi}^{B'}_k \psi^B_i \right) \cong 0 \,, \end{aligned} \tag{4.75} $$

where the corresponding ${}_{1'}\mathcal{H}_\perp$, ${}_{1'}\mathcal{H}_i$ differ from (4.71), (4.72), and (4.74) by terms proportional to ${}_1\mathcal{J}_{AB}$ or its derivatives. In more detail, Eq. (4.74) and its Hermitian conjugate can be further simplified by decomposing the 3D spin connection ${}^{(3)}\omega_i^{AA'BB'}$, contained in the covariant derivative ${}^{(3)}D_j$, into a pure bosonic part and the contorsion (A.75). The torsion-free derivative is then written as ${}^{(3s)}D_j$. This implies the following *simpler* versions of the SUSY constraints (see Exercise 4.4):

$$ {}_{1'}\mathcal{S}_A = \varepsilon^{ijk} e_{AA'i}{}^{(3s)}D_j \overline{\psi}^{A'}_k - \frac{\mathsf{k}^2}{2} \mathrm{i} p^i_{AA'} \overline{\psi}^{A'}_i \cong 0 \,, \tag{4.76} $$

$$ {}_{1'}\mathcal{S}_{A'} = \varepsilon^{ijk} e_{AA'i}{}^{(3s)}D_j \psi^A_k + \frac{\mathsf{k}^2}{2} \mathrm{i} \psi^A_i p^i_{AA'} \cong 0 \,. \tag{4.77} $$

Equations (4.76) and (4.77) correspond to the invariance of the action under left- and right-handed SUSY transformations, respectively. One relevant point is that *no* torsion terms appear in (4.76) and (4.77).

In the 2-spinor component notation, the richness of $N = 1$ SUGRA and its intrinsic symmetries provides interesting prospects for quantization. In particular, the action (4.66) is invariant under local SUSY transformations, local Lorentz transformations, and local coordinate transformations (spacetime diffeomorphisms) for the basic fields $e_\mu^{AA'}$ and ψ_μ^A. Moreover, from the second class constraints and the new Dirac brackets (instead of Poisson brackets) [25–27, 50, 61, 51, 36, 37, 33, 35], we thus obtain (see (C.93), (C.94), and (C.95))

$$\left[e_i^{AA'}(x), e_j^{BB'}(x)\right]_{\mathrm{D}} = 0 \,, \tag{4.78}$$

$$\left[e_i^{AA'}(x), p^j_{BB'}(y)\right]_{\mathrm{D}} = \varepsilon^A{}_B \varepsilon^{A'}{}_{B'} \delta_i^j \delta(x-y) \,, \tag{4.79}$$

$$\begin{aligned}\left[p^i_{AA'}(x), p^j_{BB'}(y)\right]_{\mathrm{D}} = \frac{1}{4}\big(&\varepsilon^{jln}\psi_{Bn} D_{AB'kl}\varepsilon^{ikm}\overline{\psi}_{A'm} \\ &+\varepsilon^{jln}\psi_{Am} D_{BA'lk}\varepsilon^{ikm}\overline{\psi}_{B'n}\big)\delta(x-y) \,,\end{aligned} \tag{4.80}$$

$$\left[\psi_i^A(x), \psi_j^B(y)\right]_{\mathrm{D}} = 0 \,, \tag{4.81}$$

$$\left[\psi_i^A(x), \overline{\psi}_j^{A'}(y)\right]_{\mathrm{D}} = -D_{ij}^{AA'}\delta(x-y) \,, \tag{4.82}$$

$$\left[e_i^{AA'}(x), \psi_j^B(y)\right]_{\mathrm{D}} = 0 \,, \tag{4.83}$$

$$\left[p^i_{AA'}(x), \psi_j^B(y)\right]_{\mathrm{D}} = \frac{1}{2}\varepsilon^{ikl}\psi_{Al} D^B{}_{A'jk}\delta(x-y) \,, \tag{4.84}$$

where

$$D_{ik}^{AB'} \equiv -\frac{2\mathrm{i}}{\sqrt{h}} e_k^{AC'} e_{CC'i} n^{CB'} \,. \tag{4.85}$$

The Dirac brackets above will be fundamental when considering a quantum representation. In particular, (4.78), (4.79), (4.81), and (4.82) clearly point to the tetrad and gravitino $e_i^{AA'}$, ψ_i^A as canonical variables and $p^i_{AA'}$, $\overline{\psi}^i{}_{A'}$ as the corresponding effective canonical momenta (see (4.27), (4.28), and (4.29)). The reader should note how a 2-component spinor representation is essential to obtain this Dirac bracket structure with all its implications.[23]

From (4.78), (4.79), (4.80), (4.81), (4.82), (4.83), and (4.84) and

$${}_{1'}\mathcal{H}_{AA'} \equiv -n_{AA'}\,({}_{1'}\mathcal{H}_\perp) + e^i_{AA'}({}_{1'}\mathcal{H}_i) \,, \tag{4.86}$$

[23] Although (4.80) and (4.84) are apparently somewhat odd, using [28] $\psi^A \leftarrow \varphi_i^A = h^{1/4}\psi_i^A$, the Dirac bracket would be zero, but with the price that the Hamiltonian would get more complicated, and worse, we could not obtain a wave functional $\Psi[e, \psi]$ with *independent* variables [25, 27].

using (4.75) with the corresponding ${}_{1'}\mathcal{H}_\perp$ and ${}_{1'}\mathcal{H}_i$, we have

$$\left[{}_{1'}\mathcal{S}_A(x), {}_{1'}\mathcal{S}_B(x')\right]_{\mathrm{D}} = 0\,, \tag{4.87}$$

$$\left[{}_{1'}\overline{\mathcal{S}}_{A'}(x), {}_{1'}\overline{\mathcal{S}}_{B'}(x')\right]_{\mathrm{D}} = 0\,, \tag{4.88}$$

$$\left[{}_{1'}\mathcal{S}_A(x), {}_{1'}\overline{\mathcal{S}}_{A'}(x')\right]_{\mathrm{D}} = \mathsf{k}^2\mathrm{i}\,{}_{1'}\mathcal{H}_{AA'}\delta(x,x') + \Upsilon\left[{}_{1}\mathcal{J}_{AB}, {}_{1}\overline{\mathcal{J}}_{A'B'}\right]\,, \tag{4.89}$$

where $\Upsilon\left[{}_{1}\mathcal{J}_{AB}, {}_{1}\overline{\mathcal{J}}_{A'B'}\right]$ represents terms proportional to ${}_{1}\mathcal{J}_{AB}, {}_{1}\overline{\mathcal{J}}_{A'B'}$.

4.2 Quantization of *N*=1 Supergravity

In the last section, the Hamiltonian formalism allowed us to acquire a richer perspective on the intrinsic structure (i.e., the symmetries and invariances) of $N = 1$ SUGRA. In this section, we discuss the specific features and possibilities that a canonical quantization of SUGRA can offer. This will be relevant in Chap. 5.

4.2.1 A Representation for the Fermionic Momenta

A widely used step toward a quantum formulation of $N = 1$ SUGRA in 4D spacetime is, in view of Dirac's guidelines, to associate an operator, e.g., a differential operator, with the canonical momenta corresponding to the field variables. Moreover, this quantum representation requires the classical (Dirac) brackets to be replaced by $-\mathrm{i}/\hbar$ times the commutator or anticommutator of the corresponding field operators [23, 62, 63], depending on whether we are dealing with *even* (i.e., $e^i_{AA'}$) or *odd* (i.e., ψ^i_A) elements, respectively, in a Grassmanian algebra. For the particular Dirac brackets indicated in (4.78), (4.79), (4.80), (4.81), (4.82), (4.83), and (4.84), the following operator representation of the fundamental momenta is chosen [25–27, 64]:

$$\psi^A_i \longrightarrow \overline{\psi}^{A'}_i = -\mathrm{i}\hbar D^{AA'}_{ji}\frac{\delta}{\delta\psi^A_j}\,, \tag{4.90}$$

$$e^{AA'}_i \longrightarrow p^i_{AA'} = -\mathrm{i}\hbar\frac{\delta}{\delta e^{AA'}_i} - \frac{1}{2}\mathrm{i}\hbar\varepsilon^{ijk}\psi_{Aj}D^B{}_{A'lk}\frac{\delta}{\delta\psi^B_l}\,, \tag{4.91}$$

and the quantum state will be described by a wave functional Ψ of the form

$$\Psi[e,\psi] \equiv \Psi\left[e^{AA'}_i(x), \psi^B_j(x)\right]\,. \tag{4.92}$$

The attentive reader will be wondering why we use the specific form for the quantum momenta in (4.90) and (4.91).

Note 4.6 Let us consider (4.90). Because the Dirac bracket between ψ_i^A and $\overline{\psi}_i^{A'}$ (4.82) does *not* vanish, we *cannot* choose a common basis of eigenstates for both of them. Quantum mechanically, it does indeed imply (4.90), requiring the variable ψ_j^B to be brought to the left before a derivative is applied. Obviously, ψ_i^A could have been represented by a derivative with respect to $\overline{\psi}_i^{A'}$. But this means we have instead chosen a basis for the states of the system including eigenstates of $\overline{\psi}_i^{A'}$. The wave functional would be of the form $\overline{\Psi}[e, \overline{\psi}]$, related to $\Psi[e, \psi]$ by means of functional Fourier transformations. There is, however, another approach, as we will describe in detail in Chap. 7 of Vol. II, which involves associating a matrix representation for the fermions and the corresponding momenta.

Concerning now (4.91), it happens that this form for the representation of the momenta allows the algebra of the quantum constraints to have a simpler form and specific factor ordering in the SUSY quantum constraints. This is described in some detail in the next section. In addition, it allows the Dirac brackets (4.78)–(4.84) to be satisfied quantum mechanically [25–27], and it allows an inner product with respect to which $p_i^{AA'}$ is Hermitian and ψ_A^i, $\overline{\psi}_{A'}^j$ are Hermitian adjoints.

4.2.2 Algebra of (Quantum) Constraints

Using the above representation for the canonical momenta, the constraints introduced in Sect. 4.1.4 acquire the following (quantum mechanical) operator form (indicated by hats in the usual way):

- **Quantum Lorentz Constraints.** Using (4.73) directly with (4.90) and (4.91), the quantized Lorentz constraints are found to be

$$ {}_1\widehat{\mathcal{J}}_{AB} = -\frac{i\hbar}{2}\left(e^{A'}{}_{Ba}\frac{\delta}{\delta e_a^{AA'}} + e^{A'}{}_{Aa}\frac{\delta}{\delta e_a^{BA'}} + \psi_{Ba}\frac{\delta}{\delta \psi^A{}_a} + \psi_{Aa}\frac{\delta}{\delta \psi_a^B}\right) , \quad (4.93)$$

$$ {}_1\overline{\widehat{\mathcal{J}}}_{A'B'} = -\frac{i\hbar}{2}\left(e^A{}_{B'a}\frac{\delta}{\delta e_a^{AA'}} + e^A{}_{A'a}\frac{\delta}{\delta e_a^{AB'}}\right) , \quad (4.94)$$

with

$$ {}_1\widehat{\mathcal{J}}_{AB}\Psi = 0 , \quad (4.95)$$

$$ {}_1\overline{\widehat{\mathcal{J}}}_{A'B'}\Psi = 0 , \quad (4.96)$$

corresponding to the invariance of $\Psi[e, \psi]$ under local Lorentz transformations, with the momentum operator ordered on the right.

- **Quantum Supersymmetry Constraints.** From (4.76) and (4.77), with (4.90) and (4.91), the quantized SUSY constraints are then written as

$$
{}_{1'}\widehat{\overline{\mathcal{S}}}_{A'} = \varepsilon^{ijk} e_{AA'i}\ {}^{(3s)}D_j \psi_k^A + \frac{\mathsf{k}^2}{2} \hbar \psi_i^A \frac{\delta}{\delta e_i^{AA'}} \,, \tag{4.97}
$$

$$
{}_{1'}\widehat{\mathcal{S}}_A = \mathrm{i}\hbar\, {}^{(3s)}D_i \left(\frac{\delta}{\delta \psi_i^A} \right) + \frac{\mathsf{k}^2}{2} \mathrm{i}\hbar \frac{\delta}{\delta e_i^{AA'}} \left(D_{ji}^{BA'} \frac{\delta}{\delta \psi_j^B} \right) \,, \tag{4.98}
$$

where

$$
{}_{1'}\widehat{\overline{\mathcal{S}}}_{A'} \Psi[e, \psi] = 0 \,, \tag{4.99}
$$

$$
{}_{1'}\widehat{\mathcal{S}}_A \Psi[e, \psi] = 0 \,, \tag{4.100}
$$

are the SUSY invariant properties for $\Psi[e, \psi]$, with the factor ordering in the second term of (4.97) to allow induced SUSY transformations from the constraints ${}_{1'}\widehat{\overline{\mathcal{S}}}_{A'}$ and ${}_{1'}\widehat{\mathcal{S}}_A$, within a suitable representation, i.e., $\Psi[e, \psi]$ or $\overline{\Psi}[e, \overline{\psi}]$, respectively). It should be emphasized that the factor ordering is solely determined from SUSY considerations regarding properties of the quantum states. It should also be added that, e.g., for $\psi_i p^i_{AA'}$, using (4.90) and (4.91), the resulting fermionic cubic terms like $\psi_i^B \varepsilon^{ijk} \psi_{Bj} \overline{\psi}_{A'k}$ are zero [26, 27].

- **Quantum Hamiltonian and Diffeomorphism Constraints.** The Hamiltonian and diffeomorphism constraints are written in the following (condensed) quantum mechanical form (see (4.86)):

$$
\begin{aligned}
{}_2\widehat{\mathcal{H}}_{AA'} = {} & \frac{\mathsf{k}^2}{2} \mathrm{i}\hbar^2 \psi_i^B \frac{\delta}{\delta e_j^{AB'}} \left(\varepsilon^{ilm} D_B{}^{B'}{}_{mj} D^C{}_{A'kl} \frac{\delta}{\delta \psi_k^C} \right) + n_{AA'} \frac{8\pi}{\mathsf{k}^2} U[e] \\
& - \frac{\mathsf{k}^2}{2} \mathrm{i}\hbar^2 \frac{\delta}{\delta e_j^{AB'}} \left(D^{BB'}{}_{ij} \frac{\delta}{\delta e_i^{BA'}} \right) \\
& - \frac{\mathrm{i}\hbar}{2} \varepsilon^{ijk} \left\{ \left[{}^{(3s)}D_j \psi_{Ak} \right] D^B{}_{A'li} \frac{\delta}{\delta \psi_l^B} + \psi_{Ai} \left[{}^{(3s)}D_j D^B{}_{A'lk} \frac{\delta}{\delta \psi_l^B} \right] \right\} \\
& - \mathrm{i}\hbar\, {}^{(3s)}D_i \left(\frac{\delta}{\delta e_i^{AA'}} + \frac{1}{2} \varepsilon^{ijk} \psi_{Aj} D^B{}_{A'lk} \frac{\delta}{\delta \psi_l^B} \right) \,,
\end{aligned} \tag{4.101}
$$

where $U[e] \equiv \sqrt{h}\,{}^{(3s)}R/16\pi$. The third and fourth lines constitute *new* contributions from $N = 1$ SUGRA.

Meanwhile, the attentive reader may be wondering about the following points:

- How did expression (4.101) come about, i.e., how does it relate to (4.86)?
- Does it include the Hamiltonian and diffeomorphism quantum constraints discussed in Sect. 4.1.4?

Concerning the fermionic differential operator representation (4.90), (4.91) adopted here, quantum constraints for supersymmetric quantum gravity could be established directly from the (4.71), (4.72), (4.73), and (4.74) of the $N = 1$ SUGRA action, with the resulting quantum constraints inheriting the suffix 1, e.g., being denoted by ${}_1\widehat{\mathcal{H}}_\perp$, ${}_1\widehat{\mathcal{H}}_i$, and ${}_1\widehat{S}_A$.

In a complementary way, from the other alternative (4.75), (4.76), and (4.77) for ${}_{1'}\mathcal{H}_\perp$, ${}_{1'}\mathcal{H}_i$, ${}_{1'}\mathcal{S}$, and ${}_{1'}\overline{\mathcal{S}}$, a simple procedure does lead to (4.97) and (4.98). The benefits are that it corresponds to the well known general feature of all supersymmetric theories: the commutator of a primed and an unprimed SUSY transformation yields a coordinate transformation in spacetime. This is expressed via the anticommutator that *defines*

$$\left[{}_2\widehat{\mathcal{S}}_A(x), {}_2\widehat{\overline{\mathcal{S}}}_{A'}(y)\right]_{\rm D} \equiv 4\pi G\hbar \; {}_2\widehat{\mathcal{H}}_{AA'}(x)\delta(x, y) \; , \tag{4.102}$$

where we use ${}_{1'}\widehat{\mathcal{S}}_A(x) \equiv {}_2\widehat{\mathcal{S}}_A(x)$ and ${}_{1'}\widehat{\overline{\mathcal{S}}}_{A'}(x) \equiv {}_2\widehat{\overline{\mathcal{S}}}_{A'}(x)$ as the constraints corresponding to the SUSY transformations, and the combination

$${}_2\widehat{\mathcal{H}}_{AA'} = -n_{AA'}({}_2\widehat{\mathcal{H}}_\perp) + e^i{}_{AA'}({}_2\widehat{\mathcal{H}}_i) \; , \tag{4.103}$$

where ${}_2\widehat{\mathcal{H}}_{AA'}$ *contains* the (quantum) generators of spacetime coordinate transformations, and ${}_2\widehat{\mathcal{H}}_i$ and ${}_2\widehat{\mathcal{H}}_\perp$ denote, respectively, the gravitational momentum (spatial diffeomorphisms) and Hamiltonian constraints.

Note 4.7 In addition:

- ${}_2\widehat{\mathcal{H}}_i$ represents *new* momentum constraints, differing from a quantum version of (4.72) or from (4.87), (4.88), and (4.89) by *additional* combinations of the Lorentz constraints, while ${}_2\widehat{\mathcal{H}_\perp}$ is the *new* Hamiltonian constraint, differing from a quantum version of (4.71) or from (4.87), (4.88), and (4.89) by other combinations of the Lorentz constraints [25–27, 34, 40].
- It is important to emphasize that ${}_{1'}\widehat{\mathcal{H}}_{AA'} \neq {}_2\widehat{\mathcal{H}}_{AA'}$ by terms proportional to $\mathcal{J}_{AB}$, $\overline{\mathcal{J}}_{A'B'}$, their projections, or their derivatives. However, it should in principle be possible (although it has not yet been fully demonstrated) to (a) proceed from ${}_{1'}\mathcal{H}_\perp$, ${}_{1'}\mathcal{H}_i$ to ${}_2\mathcal{H}_\perp$, ${}_2\mathcal{H}_i$, by redefining Lagrange multipliers, or to (b) project ${}_2\mathcal{H}_{AA'}$ to ${}_2\mathcal{H}_\perp$, ${}_2\mathcal{H}_i$ directly.
- It should be remarked that ${}_2\widehat{\mathcal{H}}_{AA'}$ and ${}_2\mathcal{H}_{AA'}$ are not strictly the same. ${}_2\mathcal{H}_{AA'}$ has to be computed from ${}_2\mathcal{S}_A$, ${}_2\overline{\mathcal{S}}_{A'}$.
- The quantum state $\Psi[e, \psi]$ can be found solely by using $\widehat{\mathcal{J}}_{AB}$, $\widehat{\overline{\mathcal{J}}}_{A'B'}$, ${}_2\widehat{\mathcal{S}}_A$, and ${}_2\widehat{\overline{\mathcal{S}}}_{A'}$, with the help of the algebra (4.102).
- ${}_2\widehat{\mathcal{H}_\perp}$ constitutes the normal projection of the constraint ${}_2\widehat{\mathcal{H}}_{AA'}$, retrieved from (4.103) after multiplication by and contraction with $-n^{AA'}$. This procedure of taking the anticommutator of the SUSY constraint ${}_2\widehat{\mathcal{S}}_A$ with

its Hermitian conjugate does indeed produce the much simpler form of ${}_2\widehat{\mathcal{H}}_{AA'}$ in (4.103). It thus produces a much simpler form of ${}_2\widehat{\mathcal{H}}_{\perp}$, inducing a 'many fingered' time-evolution-extended Wheeler–DeWitt equation for Ψ (see Vol. II), and also for ${}_2\widehat{\mathcal{H}}_i$, associated with the invariance of Ψ under spatial coordinate transformations within $N = 1$ SUGRA [see Sect. 4.1.3, Exercise 2.3, and (4.101)–(4.102)].

Regarding the question of whether the full quantum algebra of constraints is anomaly free, this remains an open issue (as in all approaches to canonical quantum gravity) [4]. In fact, calculations in [65] have suggested that anomalies may occur in the commutators of the SUSY constraints with $\mathcal{H}_{AA'}(x)$. A definite statement can, however, only be made after a rigorous regularization scheme has been employed.

The canonical quantization of $N = 1$ SUGRA offers plenty to explore, with a view to further clarifying the subject. In Vol. II, we briefly debate this topic, where additional discussion and investigation are welcome. It is also a sensitive issue, i.e., it involves 'conflicting' views about whether SUGRA is a finite theory, and whether canonical quantization methods could establish that [66–68, 26, 69–71].

4.2.3 Including (Super)Matter. General Formalism

This section will take the explorer gradually toward a more realistic venture, regarding the study of the very early universe (see Chap. 5). The introduction of matter, usually referred to as *supermatter*, in SUGRA leads to new and important results (see Sect. 3.5), most notably, the possibility of unifying all interactions [12] and the renormalizability of quantum gravity effects [72].

The generalization of the canonical formulation of pure $N = 1$ SUGRA to include matter fields was described in [73, 26, 60] using a 2-spinor notation. Besides its dependence on the tetrad $e^{AA'}{}_{\mu}$ and the gravitino field $\psi^{A}{}_{\mu}$, $\overline{\psi}^{A'}{}_{\mu}$, the most general $N = 1$ SUGRA theory coupled to gauged supermatter also includes:[24]

- a vector field $A^{(a)}_{\mu} \equiv v^{(a)}_{\mu}$ labelled by an index (a),
- its spin-1/2 partners $\lambda^{(a)}_{A}$, $\overline{\lambda}^{(a)}_{A'}$,
- a family of scalars ϕ^{I}, ϕ^{J^*}, and
- Their spin-1/2 partners χ^{I}_{A}, $\overline{\chi}^{\mathsf{J}^*}_{A'}$.

---

[24] See Appendix A and Sect. 3.5. We recall that the indices I and $\mathsf{J}^* \equiv \bar{\mathsf{J}}$ are Kähler indices, and that the Kähler metric herein is $\mathsf{G}_{\mathsf{IJ}^*} = \mathsf{G}_{\mathsf{I}\bar{\mathsf{J}}} = \mathsf{K}_{IJ^*}$ on the space of ϕ^{I}, ϕ^{J^*} (the Kähler manifold), where $\mathsf{K}_{\mathsf{IJ}^*}$ is a shorthand for $\partial^2 \mathsf{K}/\partial\phi^{\mathsf{I}}\partial\phi^{\mathsf{J}^*}$, with K the Kähler potential (see Note 3.8). Each index (a) corresponds to an independent (holomorphic) Killing vector field $X^{(a)}$ of the Kähler geometry. The corresponding Killing equation implies that there exist real scalar functions $D^{(a)}(\phi^I, \phi^{I^*})$ known as Killing potentials.

Its Lagrangian[25] is given, e.g., in (25.12) of [13].

The content of Sect. 4.2 is thus under extension here. The essentially new feature is that, besides the spin-3/2 gravitino, the spin-1/2 fields also yield second class constraints through the corresponding canonical momenta [33]. These are eliminated when Dirac brackets are introduced. Non-trivial Dirac brackets are made simple as follows:

- The brackets involving ${p_{AA'}}^i$, ${\psi^A}_i$, and ${\overline{\psi}^{A'}}_i$ can be simplified as in the case of pure $N = 1$ SUGRA by using the procedures of Sect. 4.1.4.
- The ϕ^{K} and ϕ^{K^*} dependence of K_{IJ^*} is responsible for unwanted Dirac brackets among χ^{I}_A, $\overline{\chi}^{\mathsf{J}^*}_A$, $\pi_{\phi_{\mathsf{L}}}$, and $\pi_{\phi^*_{\mathsf{L}}}$. In fact, defining $\pi_{\mathsf{I}A}$ and $\overline{\pi}_{\mathsf{I}^*A'}$ to be the momenta conjugate to $\chi^{\mathsf{I}A}$ and $\overline{\chi}^{\mathsf{I}^*A'}$, respectively, one has

$$\pi_{\mathsf{I}A} + \frac{\mathrm{i}h^{1/2}}{\sqrt{2}}\mathsf{K}_{\mathsf{IJ}^*}n_{AA'}\overline{\chi}^{\mathsf{J}^*A'} = 0\ ,$$

$$\overline{\pi}_{\mathsf{J}^*A'} + \frac{\mathrm{i}h^{1/2}}{\sqrt{2}}\mathsf{K}_{\mathsf{IJ}^*}n_{AA'}\chi^{\mathsf{I}A} = 0\ .$$

- We then redefine[26]

$$\hat{\chi}_{\mathsf{I}A} = h^{1/4}\mathsf{K}^{1/2}_{\mathsf{IJ}^*}\delta^{\mathsf{KJ}^*}\chi_{\mathsf{K}A}\ , \qquad \hat{\overline{\chi}}_{\mathsf{I}^*A'} = h^{1/4}K^{1/2}_{\mathsf{JI}^*}\delta^{\mathsf{JK}^*}\overline{\chi}_{\mathsf{K}^*A'}\ .$$

---

25 For later use, we recall the relevant definitions (see also Sects. 3.3.1 and 3.5):

- $\Gamma^{\mathsf{I}}_{\mathsf{JK}}$ is a Christoffel symbol derived from the Kähler metric.
- P is henceforth a complex scalar-field-dependent analytic potential energy term, which will relate to the superpotential.
- $\check{\mathrm{D}}_{\mathrm{I}} \equiv \dfrac{\partial}{\partial\phi^{\mathsf{I}}} + \dfrac{\partial\mathsf{K}}{\partial\phi^{\mathsf{I}}}$.
- $\tilde{\mathrm{D}}_\mu \equiv \partial_\mu + \omega_\mu + \dfrac{1}{4}\left(\dfrac{\partial\mathsf{K}}{\partial\phi^{\mathsf{J}}}\hat{\mathrm{D}}_\mu\phi^{\mathsf{J}} - \dfrac{\partial\mathsf{K}}{\partial\overline{\phi}^{\mathsf{J}^*}}\hat{\mathrm{D}}_\mu\overline{\phi}^{\mathsf{J}^*}\right) + \dfrac{1}{2}A^{(a)}_\mu \mathrm{Im}\, F^{(a)}$.
- $\hat{\mathrm{D}}_\mu\phi^{\mathsf{I}} \equiv \partial_\mu\phi^{\mathsf{I}} - v^{(a)}_\mu X^{\mathsf{I}(a)}\ , \qquad F^{(a)} \equiv X^{I(a)}\dfrac{\partial\mathsf{K}}{\partial\phi^{\mathsf{I}}} + \mathrm{i}D^{(a)}$.
- ${C^{\mu\nu}}_A B \equiv \dfrac{1}{4}\left(e^\mu_{AA'}e^{\nu BA'} - e^\nu_{AA'}e^{\mu BA'}\right)$.
- $\hat{f}^{(a)}_{\mu\nu} \equiv f^{(a)}_{\mu\nu} - \mathrm{i}\left\{\psi^A_{[\mu}e_{\nu]AA'}\overline{\lambda}^{(a)A'} - \overline{\psi}_{A'[\mu}e^{AA'}_{\nu]}\lambda^{(a)}_A\right\}$.

26 Here $\mathsf{K}^{1/2}_{\mathsf{IJ}^*}$ denotes a 'square-root' of the Kähler metric, obeying $\mathsf{K}^{1/2}_{\mathsf{IJ}^*}\delta^{\mathsf{KJ}^*}\mathsf{K}^{1/2}_{\mathsf{KL}^*} = \mathsf{K}_{\mathsf{IL}}$. This may be found by diagonalizing $\mathsf{K}_{\mathsf{IJ}^*}$ via a unitary transformation, assuming that the eigenvalues are all positive. One needs to assume that there is an 'identity metric' δ^{KJ^*} defined over the whole Kähler manifold.

- The brackets among $p_{AA'}{}^{i}$, $\lambda_A^{(a)}$, $\overline{\lambda}_{A'}^{(a)}$, $\hat{\chi}_A^{\mathsf{I}}$, and $\hat{\overline{\chi}}_{A'}^{\mathsf{J}^*}$ are dealt with by defining $\hat{\lambda}_A^{(a)} = h^{1/4}\lambda_A^{(a)}$ and $\hat{\overline{\lambda}}_{A'}^{(a)} = h^{1/4}\overline{\lambda}_{A'}^{(a)}$, and then going to the *time gauge*[27] [32, 33].
- All the resulting Dirac brackets are either the same as in Sect. 4.2.2 or zero, except for the nonzero fermionic brackets like

$$\left[\hat{\lambda}_A^{(a)}(x), \hat{\overline{\lambda}}^{(b)}{}_{A'}(x)\right]_{\mathrm{D}} = \sqrt{2}\mathrm{i}n_{AA'}\delta^{(a)(b)}\delta(x, x') \,, \tag{4.104}$$

$$\left[\hat{\chi}^{\mathsf{I}}{}_A(x), \hat{\overline{\chi}}^{\mathsf{J}^*}{}_{A'}(x')\right]_{\mathrm{D}} = \sqrt{2}\mathrm{i}n_{AA'}\delta^{\mathsf{I}\mathsf{J}^*}\delta(x, x') \,, \tag{4.105}$$

$$\left[\psi^A{}_i(x), \overline{\psi}^{A'}{}_j(x')\right]_{\mathrm{D}} = \frac{1}{\sqrt{2}}D^{AA'}{}_{ij}\delta(x, x') \,. \tag{4.106}$$

Regarding the full Hamiltonian, the constraints acquire *new* terms. In more detail:

- It contains arbitrary Lorentz rotations, now including the components which depend on the supermatter fields. By employing the redefinition $\mathcal{M}_{AB} \mapsto \omega_{AB0}$ and the Hermitian conjugate once again, the Lorentz contributions will yield *new* terms of the type $\psi\chi\overline{\chi}$, $\psi\lambda\overline{\lambda}$, and their Hermitian conjugates, e.g., to the supersymmetry constraints.
- The supersymmetry constraint $\overline{\mathcal{S}}_{A'}$ then takes the form[28]

$$\begin{aligned}
\overline{\mathcal{S}}_{A'} = &-\sqrt{2}\mathrm{i}e_{AA'i}\psi^A{}_j\pi^{ij} + \sqrt{2}\varepsilon^{ijk}e_{AA'i}\,{}^{3\mathrm{s}}\tilde{\mathrm{D}}_j\psi^A{}_k \\
&+\frac{1}{\sqrt{2}}\Bigg(\pi_{\mathsf{J}^*} + \frac{\mathrm{i}}{\sqrt{2}}h^{1/2}g_{\mathsf{LM}^*}\Gamma^{\mathsf{M}^*}_{\mathsf{J}^*\mathsf{N}^*}n^{BB'}\overline{\chi}^{\mathsf{N}^*}{}_{B'}\chi^{\mathsf{L}}{}_B \\
&-\frac{\mathrm{i}h^{1/2}}{2\sqrt{2}}\mathsf{K}_{\mathsf{J}^*}\mathsf{G}_{\mathsf{MM}^*}n^{BB'}\overline{\chi}^{\mathsf{M}^*}{}_{B'}\chi^{\mathsf{M}}{}_B - \frac{1}{2\sqrt{2}}\varepsilon^{ijk}\mathsf{K}_{\mathsf{J}^*}e^{BB'}{}_j\psi_{kB}\overline{\psi}_{iB'} \\
&-w_{[1]}\sqrt{2}h^{1/2}\mathsf{G}_{\mathsf{IJ}^*}\chi^{\mathsf{I}B}e_{BB'}{}^m n^{CB'}\psi_{mC}\Bigg)\overline{\chi}^{\mathsf{J}^*}{}_{A'} \\
&-\sqrt{2}h^{1/2}\mathsf{G}_{\mathsf{IJ}^*}\tilde{\mathrm{D}}_i\phi^{\mathsf{I}}\overline{\chi}^{\mathsf{J}^*}_{B'}n^{BB'}e_{BA'}{}^i + w_{[2]}\frac{\mathrm{i}}{2}\mathsf{g}_{\mathsf{IJ}^*}\varepsilon^{ijk}e_{AA'j}\psi^A{}_i\overline{\chi}^{\mathsf{J}^*B'}e_{BB'k}\chi^{\mathsf{I}B} \\
&+w_{[3]}\frac{1}{4}h^{1/2}\psi_{Ai}\left(e_{BA'}{}^i n^{AC'} - e^{AC'i}n_{BA'}\right)\mathsf{G}_{\mathsf{IJ}^*}\overline{\chi}^{\mathsf{J}^*}_{C'}\chi^{\mathsf{I}B}
\end{aligned}$$

---

[27] In this case, the tetrad component n^a of the normal vector n^μ is restricted by $n^a = \delta^a_0 \Leftrightarrow e^0{}_i = 0$. Thus, the original Lorentz rotation freedom becomes replaced by that of spatial rotations. In the time gauge, the geometry is described by the triad $e^a{}_i$ $(a = 1, 2, 3)$, and the conjugate momentum is $p_a{}^i$. Notice that

$$\pi^{ij} \equiv -\frac{1}{2}p^{(ij)} = \frac{1}{2}e^{AA'(i}p^{j)}_{AA'} = -\frac{1}{2}e^{a(i}p^{j)}_a \,,$$

where the last equality follows from the time gauge conditions.

[28] $\pi^{n(a)}$ is the momentum conjugate to $A_n^{(a)}$.

$$
\begin{aligned}
&-h^{1/2}\exp(\mathsf{K}/2)\left[2\mathsf{P}n^{A}{}_{A'}e_{AB'}{}^{i}\tilde{\psi}^{B'}{}_{i}+\mathrm{i}(\check{D}_{\mathsf{I}}\mathsf{P})n_{AA'}\chi^{\mathsf{I}A}\right]\\
&-\frac{\mathrm{i}}{\sqrt{2}}\pi^{n(a)}e_{BA'n}\lambda^{(a)B}+\frac{1}{2\sqrt{2}}\varepsilon^{ijk}e_{BA'k}\lambda^{(a)B}f_{ij}^{(a)}+\frac{1}{\sqrt{2}}h^{1/2}D^{(a)}n^{A}{}_{A'}\lambda^{(a)}{}_{A}\\
&+w_{[4]}\frac{1}{4}h^{1/2}\psi_{Ai}\left(e_{BA'}{}^{i}n^{AC'}-e^{AC'i}n_{BA'}\right)\overline{\lambda}^{(a)}_{C'}\lambda^{(a)B}\\
&-\frac{\mathrm{i}}{4}h^{1/2}n^{BB'}\lambda^{(a)}{}_{B}\overline{\lambda}^{(a)}{}_{B'}\mathsf{K}_{\mathsf{J}^*}\overline{\chi}^{\mathsf{J}^*}{}_{A'}\ ,
\end{aligned}
\tag{4.107}
$$

where $\lambda^{(a)}_{A}$, $\overline{\lambda}^{(a)}_{A'}$, $\chi_{\mathsf{I}A}$, and $\overline{\chi}_{\mathsf{I}^*A'}$ should be redefined as indicated above. The other supersymmetry constraint $\mathcal{S}_A$ is just the Hermitian conjugate of (4.107). The $w_{[i]}$, $i = 1, 2, 3, 4$ denote numerical coefficients which correspond to inclusion of the terms $\psi\chi\overline{\chi}$, $\psi\lambda\overline{\lambda}$ and their Hermitian conjugates in the supersymmetry constraints via $\omega_{AB0}J^{AB}$ and its Hermitian conjugate in the full Hamiltonian.[29]

- The total Hamiltonian now includes $v_0^{(a)}\mathcal{Q}_{(a)}$, where $\mathcal{Q}_{(a)}$ is the generator of gauge invariance. The gauge generator $\mathcal{Q}^{(a)}$ is given classically by

$$
\begin{aligned}
\mathcal{Q}^{(a)} = &-\partial_n\pi^{n(a)}-\mathsf{f}^{(a)(b)(c)}\pi^{n(b)}v_n^{(c)}+\pi_{\mathsf{I}}X^{\mathsf{I}(a)}+\pi_{\mathsf{I}^*}X^{\mathsf{I}^*(a)}\\
&+\sqrt{2}\mathrm{i}h^{1/2}\mathsf{K}_{\mathsf{MI}^*}n^{AA'}X^{\mathsf{J}^*(a)}\Gamma^{\mathsf{I}^*}_{\mathsf{J}^*\mathsf{N}^*}\overline{\chi}^{\mathsf{N}^*}_{A'}\chi^{\mathsf{M}}_{A}\\
&-\sqrt{2}\mathrm{i}h^{1/2}n^{AA'}\overline{\lambda}^{(b)}_{A'}\left[\mathsf{f}^{(a)(b)(c)}\lambda^{(c)}_{A}+\frac{1}{2}\mathrm{i}\,\mathrm{Im}\,F^{(a)}\lambda^{(b)}_{A}\right]\\
&+\sqrt{2}\mathrm{i}h^{1/2}n^{AA'}\mathsf{K}_{\mathsf{IJ}^*}\overline{\chi}^{\mathsf{J}^*}_{A'}\left[\frac{\partial X^{\mathsf{I}(a)}}{\partial\phi^{\mathsf{J}}}\chi^{\mathsf{J}}_{A}+\frac{1}{2}\mathrm{i}\,\mathrm{Im}\,F^{(a)}\chi^{\mathsf{I}}_{A}\right]\\
&-\frac{\mathrm{i}}{\sqrt{2}}\,\mathrm{Im}\,F^{(a)}\varepsilon^{ijk}\overline{\psi}_{iA'}e^{AA'}{}_{j}\psi_{Ak}\ ,
\end{aligned}
\tag{4.108}
$$

  where $\mathsf{f}^{(a)(b)(c)}$ are the structure constants of the Kähler isometry group.

We now expect to obtain the correct transformation properties for the physical fields under supersymmetry transformations, using the brackets, e.g.,

$$
\delta_\varepsilon\psi_i^A \equiv \left[\overline{\varepsilon}_{A'}\overline{S}^{A'},\psi_i^A\right]_{\mathrm{D}}\ ,
$$

where $\overline{\varepsilon}^{A'}$ is a constant spinor parametrizing the supersymmetric transformation. In fact, this was *not* possible for some fields, while using the explicit form of the supersymmetry constraints present in [73, 26]. The reasons are as follows:

- On the one hand, the matter terms in the Lorentz constraints $\mathcal{J}_{AB}$, $\overline{\mathcal{J}}_{A'B'}$ were *not* included in the supersymmetry constraints. *If* included (see (4.107)), the $w_{[i]}$, $i = 1, 2, 3, 4$ coefficients (then to be determined!) would be different.

---

29 In particular, notice again the form of the extended spatial derivatives:

$$
{}^{(3s)}\tilde{\mathrm{D}}_j\psi^{A}{}_{k}=\partial_j\psi^{A}{}_{k}+{}^{(3s)}\omega^{A}{}_{Bj}\psi^{B}{}_{k}+\frac{1}{4}\left(\mathsf{K}_{\mathsf{K}}\hat{\mathrm{D}}_j\phi^{\mathsf{K}}-\mathsf{K}_{\mathsf{K}^*}\hat{\mathrm{D}}_j\phi^{\mathsf{K}^*}\right)\psi^{A}{}_{k}+\frac{1}{2}v_j^{(a)}\mathrm{Im}\,F^{(a)}\psi^{A}{}_{k}\ ,
$$

where $\hat{\mathrm{D}}_i\phi^{\mathsf{K}} = \partial_i\phi^{\mathsf{K}} - v_i^{(a)}X^{\mathsf{K}(a)}$, and $X^{\mathsf{K}(a)}$ is the a th Killing (Kähler) vector field [13].

- On the other hand, expressions *only valid* in pure $N = 1$ SUGRA were employed to simplify the supersymmetry constraints with supermatter. In particular, the expressions for $\mathcal{S}_A = 0$, $\overline{\mathcal{S}}_{A'} = 0$ in pure $N = 1$ SUGRA were used to rewrite the spatial covariant derivative ${}^{(3)}D_i$ in terms of its torsion-free part ${}^{(3s)}D_i$ and remaining terms which include the contorsion.
- Hence, *further work* may still be required in order to produce the description of $N = 1$ SUGRA with supermatter. Once obtained, we can proceed and extend from Chap. 4 of Vol. II, writing $\mathcal{H}_\perp^T = -n^{AA'}\mathcal{H}_{AA'}^T$, where $\mathcal{H}_\perp^T = \mathcal{H}_\perp + \mathcal{H}_\perp^{\text{matter}}$, $\mathcal{H}_i^T = e^{AA'i}\mathcal{H}_{AA'i}^T$, and $\mathcal{H}_i^T = \mathcal{H}_i + \mathcal{H}_i^{\text{matter}}$, whence it should be possible to investigate the implications of supermatter (see Sect. 3.5) in the semiclassical limit.

Summary and Review. Before moving on to the main core of SQC (see Part III of this book), let us have a look at some of the main points and elements from the Hamiltonian formulation and the canonical quantization of the full theory of $N = 1$ SUGRA:

1. Why is it of relevance to investigate $N = 1$ SUGRA from a Hamiltonian point of view [Sect. 4.1]?
2. How does the torsion become explicit in terms of the gravitino [Sect. 4.1.2]?
3. What changes are brought about by using Dirac brackets [Sect. 4.1.2]?
4. Review the way the secondary (first class) constraints are retrieved. How is the Lorentz constraint included and what is the Teitelboim procedure [Sects. 4.1.2 and 4.1.3]?
5. How do the 'spacetime' solutions in $N = 1$ SUGRA differ from the spacetime solutions in conventional geometrodynamics [Sect. 4.1.3]?
6. Review the Hamiltonian formulation for the 2-spinor notation. Are there advantages with regard to the structure of the Dirac brackets [Sect. 4.1.4]?
7. How are the Hamiltonian and momentum constraints extracted [Sect. 4.2.2]?

Problems

4.1 Secondary Constraints and Dirac Brackets in $N = 1$ SUGRA

Apply the Dirac method [36, 37] (see Appendix B) to establish the brackets (4.27)–(4.29).

4.2 Constraints as Square-Root and SUSY

Investigate how the Dirac and Klein–Gordon equations entail SUSY when fermions are included in a square-root formulation [74–76].

4.3 Expressing $n^{AA'}$ in Terms of $e_i^{AA'}$

Using the contents of Sects. 4.1.2 and A.4, relate $n^{AA'}$ with $e_i^{AA'}$ by

$$n^{AA'} = \frac{\mathrm{i}}{3\sqrt{h}} \varepsilon^{ijk} e_i^{AB'} e_{BB'j} e_k^{BA'} . \tag{4.109}$$

4.4 Simplifying the Supersymmetry Constraints

Obtain the simplified forms (4.76) and (4.77) from (4.74).

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Chapter 5
SQC *N*=1 SUGRA (Fermionic Differential Operator Representation)

This is a very important chapter. It may seem rather long and perhaps over-detailed, but the reader should have no doubt that it is all quite necessary. All journeys of exploration, including those involved with research, have to start somewhere, and many steps may be needed before an interesting vantage point is reached. SQC is such a case and this chapter is part of the route toward that goal. A significant amount of the results and research in SQC have been produced within the methodology we will describe here (see [1], but also [2–6]). It follows directly from the reduction of 4D canonical quantum $N = 1$ SUGRA (indirectly related to a higher dimensional superstring theory) to a cosmological minisuperspace with time-dependent (1D) variables and *momenta represented by differential operators* of the canonically conjugate variables [7, 8].

Let us be more precise. Such quantum mechanical correspondence applies to *both* bosonic and fermionic degrees of freedom, and we will therefore employ a *differential operator representation* for the fermions. The important feature is that it is fully consistent with a Dirac quantum mechanical description within the structure of (Dirac) classical brackets (see Chap. 4 and Appendix B).

In this chapter, we present cosmological scenarios within $N = 1$ SUGRA, and in Chap. 6 we deal with SUSY cosmologies retrieved from the bosonic sectors of superstring theory. However, there is another methodology in SQC, entailing specific differences in method *and* results, viz., choosing a matrix representation for the fermions which complies with the Dirac bracket algebra (see Chap. 7 of Vol. II). However, there is no compelling reason from a fundamental point of view for these approaches to quantization with fermionic variables to be physically equivalent within the framework of SQC (and nor is there any reason to prefer one over the other). In fact, a clear equivalence or relation between them is yet to be fully established, explaining clearly to what extent if any they are connected, and in what way.

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At this point the reader may be wondering again about the relevance of SQC,[1] and in particular what could possibly be the interest in studying a less elaborate configuration like a minisuperspace, for example. We should ask what it could contribute to a more complete insight into the very early universe, where quantum mechanical *and* supersymmetry features would be mandatory.

The fact is that spatially homogeneous minisuperspace models have indeed proved to be a very valuable tool in supergravity theories. In particular, the study of minisuperspaces has led to important and interesting results, pointing to unforeseen but useful lines of research in the canonical quantization of $N = 1$ SUGRA. Most of these features had not been previously contemplated in more complicated situations, and research in SQC has indeed revealed new directions for exploration. Moreover, we hope that some of the results present in the minisuperspace sector will hold in the full theory. In essence, minisuperspace cosmology with both quantum mechanical and supersymmetric features may, on the one hand, provide *lateral* perspectives on how to address far more complicated issues in the full theory (it may be easier to solve a problem in a finite-dimensional setting than in an infinite one), and on the other hand, indicate boundary limits for the theory to comply with. Of course, overall consistency is required (see, e.g., [9–11]).

Throughout this chapter we will study in some detail the canonical quantization of supersymmetric minisuperspaces extracted directly from $N = 1$ SUGRA. In Sect. 5.1, FRW cosmologies are discussed, and in Sect. 5.2, we proceed to extend (some) of the previous steps into the domain of Bianchi models. It is edifying to employ Bianchi type IX and $k = +1$ (closed) FRW models, and we will often do so, also describing the generalization of the results to Bianchi class A models.

5.1 FRW Models

FRW minisuperspaces are the simplest to investigate in SQC. We opt to follow a bottom-up approach throughout this section. It will provide a view of how SQC research may have proceeded, and also the methods that are applied and the results that can be retrieved. This will then be extended to Bianchi models in Sect. 5.2, where more spatial degrees of freedom and corresponding fermionic variables induce extra sectors in the wave function of the universe, with their accompanying difficulties. By learning the details of FRW SQC methods and their consistency with the corresponding purely bosonic quantum cosmology, the reader will acquire the confidence and broadness of perspective to address other problems, some still open to investigation (see Chaps. 7 and 8).

---

[1] Consult Sects. 4.1 and 4.2 of Vol. II for the canonical quantization of the full theory. There we identify the characteristics of physical states [12, 4, 13] (Sect. 4.1), discussing the route to a semiclassical perspective [14] and how one might investigate (cosmologically) testable directions (Sect. 4.2).

We start by reviewing the essentials of FRW minisuperspace reduction in SQC in Sect. 5.1.1, and then proceed in a crescendo from the pure vacuum (no matter) case in Sect. 5.1.2, to the inclusion of a cosmological constant in Sect. 5.1.3, assessing what SQC can provide with regard to asymptotic (anti-) de Sitter states and pointing to a difficulty in SQC and SUGRA/superstring theory in general, before adding scalar and vector fields in Sects. 5.1.4 and 5.1.5, respectively. By the end of this section, the reader will hopefully have acquired the confidence and experience to venture into other more complicated settings, e.g., spatial anisotropies in Sect. 5.2 or even inhomomogeneous perturbations in Sect. 10.1 of Vol. II, eventually addressing other problems, some of which once again remain open to investigation (see Chaps. 7 and 8).

5.1.1 FRW Minisuperspace Reduction Ansätse

Let us begin by discussing the reduction of 4D $N = 1$ SUGRA to a 1D (purely time-dependent) FRW cosmological model. The FRW ansätze is required to be consistent with supersymmetric, Lorentz, and general coordinate transformations, in order to lead to a minisuperspace that will inherit invariance under local time translations, supersymmetry, and Lorentz transformations. With regard to $k = +1$ FRW models obtained from pure $N = 1$ SUGRA, a rather *specific* ansätze for the gravitational and gravitino fields was employed in [15–17] (see also [2]). The quantum constraint equations are very simple and, in particular, the *no-boundary* (Hartle–Hawking) [18] and *wormhole* (Hawking–Page) [19] wave functions are found.

Concerning the presence of matter (see Sects. 5.1.4 and 5.1.5), a scalar supermultiplet (see Sect. 3.2.3), constituted by complex scalar fields, ϕ, $\overline{\phi}$ and their spin-1/2 partners χ_A and $\overline{\chi}_{A'}$ was considered in [15, 20, 21, 16, 17, 22–24] for FRW models. A vector supermultiplet (see Sect. 3.3.2), formed by gauge vector fields $v_\mu^{(a)} \equiv A_\mu^{(a)}$ and fermionic partners $\lambda_A^{(a)}, \overline{\lambda}_{A'}^{(a)}$, was added in [25–27].

In the following sections, we describe in some detail what features are chosen for $e^\mu_{AA'}$, ψ^μ_A, and why.

Tetrad and Gravitino Ansätze

Closed ($k = +1$) FRW universes have S^3 spatial sections. The tetrad of the 4D theory, $e^{AA'}_\mu = e^a_\mu \sigma_a^{AA'}$ in 2-spinor notation, which represents the gravitational field, can be taken as

$$e_{a\mu} = \begin{bmatrix} \mathcal{N}(\tau) & 0 \\ 0 & a(\tau) e_{\hat{a}i} \end{bmatrix}, \qquad e^{a\mu} = \begin{bmatrix} \mathcal{N}(\tau)^{-1} & 0 \\ 0 & a(\tau)^{-1} e^{\hat{a}i} \end{bmatrix}, \tag{5.1}$$

where $\hat{a}$ and i run from 1 to 3, $e_{\hat{a}i}$ is a basis of left-invariant 1-forms on the unit sphere S^3 whose volume we take here to be $\varsigma^2 \equiv 2\pi^2$, and $\mathcal{N}(t)$, $a(t)$,

$\sigma_a^{AA'}(A, A' = 0, 1)$ denote the lapse function, scale factor, and Infeld–Van der Warden symbols, respectively (see Appendix A).

This ansatz reduces the number of degrees of freedom provided by $e_{AA'\mu}$. If supersymmetry invariance is to be retained, then we also need an ansatz for $\psi^A{}_\mu$ and $\overline{\psi}^{A'}{}_\mu$, which also reduces the number of fermionic degrees of freedom. We take $\psi^A{}_0$ and $\overline{\psi}^{A'}{}_0$ to be functions of time alone. We further take[2]

$$\psi^A{}_i = e^{AA'}{}_i \overline{\psi}_{A'} , \qquad \overline{\psi}^{A'}{}_i = e^{AA'}{}_i \psi_A , \tag{5.2}$$

where we introduce the new spinors ψ_A and $\overline{\psi}_{A'}$ which are functions of time alone.[3]

> *Note 5.1* These expressions are a direct consequence of assuming an FRW geometry and the necessary condition for supersymmetry invariance to be retained. In fact, the above ansätze preserves the form of the tetrad under a *suitable combination* of supersymmetry, Lorentz, and local coordinate transformations.

From the variations of $e^\mu_{AA'}$ and ψ^μ_A under supersymmetry, Lorentz, and local coordinate transformations (see Appendix A and Sect. 4.1.4), we can write

$$\begin{aligned}\delta e_i^{AA'} &= \left[-\mathcal{N}^{AB} + a^{-1}\xi^{AB} + \mathrm{ik}\varepsilon^{(A}\psi^{B)}\right] e_B{}^{A'}{}_i \\ &\quad + \left[-\overline{\mathcal{N}}^{A'B'} + a^{-1}\overline{\xi}^{A'B'} + \mathrm{ik}\overline{\varepsilon}^{(A'}\overline{\psi}^{B')}\right] e^A{}_{B'i} \\ &\quad + \frac{\mathrm{ik}}{2}\left(\varepsilon_C \psi^C + \overline{\varepsilon}_{C'}\overline{\psi}^{C'}\right) e^{AA'}{}_i , \end{aligned} \tag{5.3}$$

where ξ^{AB}, N^{AB}, ε^A and their Hermitian conjugates parametrize local coordinate, Lorentz, and supersymmetry transformations, respectively. Notice then that the ansatz for the tetrad is preserved, i.e.,

$$\delta e_i^{AA'} \equiv \mathsf{P}_1\left[e_\mu^{AA'}, \psi_i^A\right] e_i^{AA'} ,$$

where P_1 is a functional, provided that the relations

$$\mathcal{N}^{AB} - a^{-1}\xi^{AB} - \mathrm{ik}\varepsilon^{(A}\psi^{B)} = 0 , \tag{5.4}$$

$$\overline{\mathcal{N}}^{A'B'} - a^{-1}\overline{\xi}^{A'B'} - \mathrm{ik}\overline{\varepsilon}^{(A'}\overline{\psi}^{B')} = 0 , \tag{5.5}$$

---

[2] It is also important to stress that auxiliary fields are also required to balance the number of fermionic and bosonic degrees of freedom. However, these auxiliary fields can be neglected in the end (see [15, 2, 21, 16, 17] for more details).

[3] This means that we truncate the general decomposition (see Sect. 5.2) $\psi^A{}_{BB'} = e_{BB'}{}^i \psi^A{}_i$ in (5.191) at the spin-1/2 mode level, i.e., with $\beta^A = 3n^{AA'}\overline{\psi}_{A'}/4 \sim \overline{\psi}^A$.

between the generators of Lorentz, coordinate, and supersymmetry transformations are satisfied.

The ansatz for the fields $\psi^A{}_i$ and $\overline{\psi}^{A'}{}_i$ should also be preserved under the same combination of transformations, in the presence of (5.3) and (5.5). Hence (see Exercise 5.1),

$$\begin{aligned}\delta\psi_i^A = & \frac{\mathrm{ik}}{4}\varepsilon^A\psi^B\overline{\psi}^{B'}e_{BB'i} + a^{-1}\overline{\xi}^{A'B'}e^A{}_{B'i}\overline{\psi}_{A'} \\ & + \left[\frac{2}{\mathrm{k}}\left(\frac{\dot{a}}{a\mathcal{N}}+\frac{\mathrm{i}}{a}\right) - \frac{\mathrm{ik}}{2N}\left(\psi_F\psi^F{}_0 + \overline{\psi}_{F'0}\overline{\psi}^{F'}\right)\right]n_{BA'}e^{AA'}{}_i\varepsilon^B ,\end{aligned} \tag{5.6}$$

and its Hermitian conjugate. The ansatz for $\psi^A{}_i$ is then preserved, i.e.,

$$\delta\psi_i^A \simeq \mathsf{P}_2\left[e_\mu^{AA'}, \psi_i^A\right]e_i^{AA'}\overline{\psi}_{A'} ,$$

where P_2 is a functional, if we impose the additional restriction

$$\psi^B\overline{\psi}^{B'}e_{BB'i} = 0 , \tag{5.7}$$

and take $\overline{\xi}^{A'B'} = 0$.

Hamiltonian Formulation and Constraints

Subsequently, the integration over the spatial hypersurfaces leads to a cosmological minisuperspace characterized by an $N = 4$ *local* supersymmetry and time-invariance reparametrization. Clearly, it also involves the *elimination* of *spatial indices by contraction*. In the end, we get an effective Lagrangian where only scalar functions, e.g., a cosmological scale factor $a(t)$, are present together with fermionic variables with *only* spinorial indices (see the discussion in [24]). By imposing the above symmetry conditions (5.1) and (5.2), we obtain a 1D mechanical model depending only on t.

In more concrete terms, inserting the ansätze (5.1) and (5.2) into the action of $N = 1$ SUGRA in a 4D spacetime (4.66), we obtain (after integration over the spatial hypersurfaces), the following FRW reduced Lagrangian:

$$L_{\mathrm{SUSY}}^{\mathrm{FRW}} \equiv L_{\mathrm{free}} , \tag{5.8}$$

where L_{free} includes the kinematical and curvature-induced terms, and is given by

$$
\begin{aligned}
L_{\text{free}} = & -\frac{3a}{\mathcal{N}}\left[\dot{a} - \frac{\mathrm{i}}{4}\left(\overline{\psi}_{0A'}\overline{\psi}^{A'} - \psi^{A}\psi_{0A}\right)\right]^2 \\
& -\mathrm{i}\frac{3}{2}a^3 n_{AA'}\psi^{A}\overline{\psi}^{A} - \mathrm{i}\frac{3}{2}a^3 n_{AA'}\overline{\psi}^{A}\dot{\psi}^{A} + 3\mathcal{N}a \\
& +\frac{3}{2}a^2\left(\overline{\psi}_{0A'}\overline{\psi}^{A'} + \psi^{A}\psi_{0A}\right) - \frac{3}{2}\mathcal{N}a^2 n^{AA'}\psi_{A}\overline{\psi}_{A'} \\
& +\frac{3}{16}a^3 n^{AA'}\left(\overline{\psi}_{0A'}\psi_{A}\overline{\psi}^{B'}\overline{\psi}_{B'} + \psi_{0A}\overline{\psi}_{A'}\psi^{B}\psi_{B}\right) .
\end{aligned} \tag{5.9}
$$

This process implies that the bosonic and fermionic variables will satisfy specific constraints (see Sect. 4.2). There are first class constraints (see Appendix B) obtained from the Lagrange multipliers in the (dimensionally reduced) effective Hamiltonian . Classically, these constraints vanish, forming an algebra, and are functions of the basic dynamical variables [2, 21, 16, 17].

The complete Hamiltonian of this FRW minisuperspace will have the usual form

$$
H = \mathcal{N}\mathcal{H} + \psi_0^A \mathcal{S}_A + \overline{\mathcal{S}}_{A'}\overline{\psi}_0^{A'} + \mathcal{M}^{AB}\mathcal{J}_{AB} + \overline{\mathcal{M}}^{A'B'}\overline{\mathcal{J}}_{A'B'} , \tag{5.10}
$$

where ψ_0^A and $\overline{\psi}_0^{A'}$ together with $\mathcal{M}^{AB}$ and $\overline{\mathcal{M}}^{A'B'}$ are Lagrange multipliers. Moreover, $\mathcal{H}_{\perp} \equiv \mathcal{H}$ represents the Hamiltonian constraint associated with local time transformations, while $\mathcal{S}_A$, $\overline{\mathcal{S}}_{A'}$, and $\mathcal{J}_{AB}$, $\overline{\mathcal{J}}_{A'B'}$ denote the supersymmetry and Lorentz constraints, respectively, associated with local $N = 4$ supersymmetry and Lorentz transformations. The complete Hamiltonian constraint can be written [24]

$$
\mathcal{H}^{\text{FRW}}_{\text{SUSY}} = \mathcal{H}_{\text{free}} , \tag{5.11}
$$

with

$$
\mathcal{H}_{\text{free}} \equiv -\frac{\pi_a^2}{12a} - 3a + \frac{3}{2}a^2 n^{AA'}\psi_A\overline{\psi}_{A'} . \tag{5.12}
$$

Note 5.2 The following points should be considered:

- Equation (5.7) suggests that $\psi_A\overline{\psi}_{A'} \sim n_{AA'}$ and basically constitutes a reduced form of the Lorentz constraint in the full theory. It is present in the two equivalent forms:

$$
\mathcal{J}_{AB} = \psi_{(A}\overline{\psi}^{B'}n_{B)B'} = 0 , \tag{5.13}
$$

$$
\overline{\mathcal{J}}_{A'B'} = \overline{\psi}_{(A'}\psi^{B}n_{BB')} = 0 . \tag{5.14}
$$

- By requiring that the constraint (5.14) be preserved under the same combination of transformations as used above, one finds equations that are satisfied provided that the supersymmetry constraints $\mathcal{S}_A = 0$ and $\overline{\mathcal{S}}_{A'} = 0$ hold. By further requiring that the supersymmetry constraints be preserved, one finds additionally that the Hamiltonian constraint $\mathcal{H} = 0$ should hold.

- The supersymmetry and Hamiltonian constraints read, in the pure case (see Exercise 5.2):

$$\mathcal{S}_A = \psi_A \pi_a - 6\mathrm{i}a\psi_A + \frac{\mathrm{i}}{2a} n_A^{\ E'} \psi^E \psi_E \overline{\psi}_{E'} \ , \tag{5.15}$$

$$\overline{\mathcal{S}}_{A'} = \overline{\psi}_{A'} \pi_a + 6\mathrm{i}a\overline{\psi}_{A'} - \frac{\mathrm{i}}{2a} n_E^{\ A'} \overline{\psi}^{E'} \psi_E \overline{\psi}_{E'} \ , \tag{5.16}$$

$$\mathcal{H} = -a^{-1}(\pi_a^2 + 36a^2) + 12a^{-1} n^{AA'} \psi_A \overline{\psi}_{A'} \ . \tag{5.17}$$

For the gravitino (effective) variable ψ_A, their canonical momenta produce (second class) constraints. These are eliminated when Dirac brackets are introduced [28–30], instead of the original Poisson brackets (see Sect. 4.2 and Appendix B). It is convenient to redefine ψ_A in order to simplify the Dirac brackets (see below):

$$\hat{\psi}_A \equiv \frac{\sqrt{3}}{2^{1/4}} \varsigma a^{3/2} \psi_A \ , \qquad \hat{\overline{\psi}}_{A'} \equiv \frac{\sqrt{3}}{2^{1/4}} \varsigma a^{3/2} \overline{\psi}_{A'} \ , \tag{5.18}$$

where the conjugate momenta are

$$\pi_{\hat{\psi}_A} = \mathrm{i} n_{AA'} \hat{\overline{\psi}}^{A'} \ , \qquad \pi_{\hat{\overline{\psi}}_{A'}} = \mathrm{i} n_{AA'} \hat{\psi}^A \ , \tag{5.19}$$

and the corresponding Dirac bracket becomes

$$\left[\hat{\psi}_A, \hat{\overline{\psi}}_{A'} \right]_{\mathrm{D}} = \mathrm{i} n_{AA'} \ . \tag{5.20}$$

Furthermore, we have

$$[a, \pi_a]_{\mathrm{D}} = 1 \ , \tag{5.21}$$

and the rest of the brackets are zero. Hereafter, we drop the hat over the variables. Additionally, it is simpler to describe the theory using only *unprimed spinors*, and to this end, it is helpful to further define

$$\overline{\psi}_A \equiv 2n_A^{\ B'} \overline{\psi}_{B'} \ , \tag{5.22}$$

whereupon the new Dirac brackets are

$$[\psi_A, \overline{\psi}_B]_{\mathrm{D}} = \mathrm{i}\varepsilon_{AB} \ . \tag{5.23}$$

The rest of the brackets remain unchanged.

Note 5.3 Other redefinitions of the dynamical fields are relevant. We can merely let

$$a \longmapsto \frac{\mathsf{k}}{\varsigma} a \,, \qquad \psi^A \longmapsto \sqrt{\frac{2}{3}} \frac{\varsigma^{1/2}}{(\mathsf{k}a)^{3/2}} \psi^A \,,$$

and

$$\overline{\psi}^{A'} \longmapsto \sqrt{\frac{2}{3}} \frac{\varsigma^{1/2}}{(\mathsf{k}a)^{3/2}} \overline{\psi}^{A'} \,,$$

but we can further include the constraint $\mathcal{J}_{AB} = 0$ by adding suitable $\mathcal{M}'^{AB}\mathcal{J}_{AB}$ to the Lagrangian, where $\mathcal{M}'^{AB}$ are redefined Lagrange multipliers (see the Teitelboim preocedure [31] in Sect. 4.2). In order to achieve the simplest form of the generators and their Dirac brackets, the following redefinitions of the non-dynamical variables $\mathcal{N}$, ${\psi^A}_0$, ${\overline{\psi}^{A'}}_0$, and $\mathcal{M}^{AB}$ are made (see Exercise 5.2):

$$\hat{\mathcal{N}} \equiv \frac{\varsigma \mathcal{N}}{12\mathsf{k}} \,, \qquad \rho^A \equiv \frac{\mathrm{i}(\mathsf{k}\varsigma)^{1/2}}{2\sqrt{6}a^{1/2}} {\psi^A}_0 + \frac{\mathrm{i}\varsigma\mathcal{N}}{12\mathsf{k}a^2} n^{AA'} \overline{\psi}_{A'} \,, \tag{5.24}$$

$$\mathcal{M}'^{AB} \equiv \mathcal{M}^{AB} - \frac{(\mathsf{k}\varsigma)^{1/2}}{3\sqrt{6}a^{3/2}} \psi^{(A}{}_0 \psi^{B)} - \frac{2(\mathsf{k}\varsigma)^{1/2}}{3\sqrt{6}a^{3/2}} \overline{\psi}^{(A'}{}_0 \overline{\psi}^{B')} n^A{}_{A'} n^B{}_{B'} , \tag{5.25}$$

and Hermitian conjugates. In the end, the full set of constraints then take the rather *simple* form (see [16, 17] for more details)

$$\mathcal{S}_A = \psi_A \pi_a - 6\mathrm{i}a\psi_a \,, \tag{5.26}$$

$$\mathcal{H} = -a^{-1}(\pi_a^2 + 36a^2) \,, \tag{5.27}$$

$$\overline{\mathcal{S}}_{A'} = \overline{\psi}_{A'}\pi_a + 6\mathrm{i}a\overline{\psi}_{A'} \,, \tag{5.28}$$

$$\mathcal{J}_{AB} = \psi_{(A}\overline{\psi}^{B'} n_{B)B'} \,. \tag{5.29}$$

The presence of the free parameters ρ_A and $\overline{\rho}_{A'}$ shows that this model has $N = 4$ local supersymmetry in one dimension. The subsequent algebra and corresponding symmetries are identified from the reduced minisuperspace model.

5.1.2 FRW Quantum States for the Vacuum Case

Quantum mechanically, the Dirac brackets are replaced by anticommutators or commutators as follows (taking units with $\hbar = 1$ henceforth):

$$\{\psi_A, \overline{\psi}_B\} \equiv -\mathrm{i}\varepsilon_{AB} \;, \tag{5.30}$$

$$[a, \pi_a] \equiv 1 \;. \tag{5.31}$$

As we adopt the differential operator representation for *both* bosonic and fermionic conjugate momenta, as the Dirac brackets imply, we can thus choose (ψ_A, a) to be the coordinates of the configuration space and $(\overline{\psi}_A, \pi_a)$ to be the momentum operators in this representation. Regarding a quantum mechanical description (see Sect. 4.2.1), the momentum variables are subsequently represented by

$$\overline{\psi}_A \rightarrow \frac{\partial}{\partial \psi^A} \;, \qquad \pi_a \rightarrow -\mathrm{i}\frac{\partial}{\partial a} \;. \tag{5.32}$$

Thus, at a canonical quantization level, the supersymmetry constraints have the form (see (5.26) and (5.28))

$$\mathcal{S}_A = -\frac{1}{2\sqrt{3}} a \psi_A \frac{\partial}{\partial a} - \sqrt{3} a^2 \psi_A \;, \tag{5.33}$$

$$\overline{\mathcal{S}}_A = \frac{1}{2\sqrt{3}} a \frac{\partial}{\partial a} \frac{\partial}{\partial \psi^A} - \sqrt{3} a^2 \frac{\partial}{\partial \psi^A} \;. \tag{5.34}$$

Note 5.4 A significant point should be stressed. *Both* of the supersymmetry constraints are *linear* with respect to the conjugate momenta of *any* bosonic variable. It can therefore be expected that a set of first-order differential equations will be obtained, thus constituting an alternative to using only the Wheeler–DeWitt equation (strict bosonic quantum cosmology).

The Lorentz constraint associated with the field variables ψ_A and $\overline{\psi}_{A'}$ acquires the form

$$\mathcal{J}_{AB} = \psi_{(A} \overline{\psi}_{B)} = 0 \;, \tag{5.35}$$

where use of (5.32) brings in the differential operator

$$\mathcal{J}_{AB} = \psi_{(A} \frac{\partial}{\partial \psi^{B)}} \;. \tag{5.36}$$

An ansatz for the wave function of the universe can then be proposed by taking the Lorentz constraint in the differential operator form presented above. It admits as a possible solution an expression constituted by a partition into fermionic sectors, each being associated with a specific bosonic amplitude (functional of the coordinate variables a) as follows:

$$\Psi^{\mathrm{FRW}}_{\mathrm{SUSY}} = \mathsf{A}(a) + \mathsf{B}(a) \psi^F \psi_F \;. \tag{5.37}$$

In other words, we can conclude that $\mathcal{J}^{AB}\Psi = 0$ *implies* that Ψ can be written as $\Psi = \mathsf{A} + \mathsf{B}\psi_A\psi^A$, where A and B depend only on a.

Note 5.5 But the careful researcher should also remark that the quantum mechanical representation of the momenta by differential operators in the bosonic *and* fermionic conjugate momentum variables does provide a simple solution to the Lorentz constraints in our specific FRW minisuperspace scenario. However, we should stress that this aspect is a *strict consequence* of our FRW minisuperspace construction, leading to the Lagrangian or Hamiltonian where only spinorial indices are present!

When investigating anisotropic Bianchi models, the situation changes drastically. As pointed out in Chap. 7 of Vol. II, some *care must be employed* when dealing with the Lorentz invariance of physical states. *Both* spatial and spinorial indices are available to construct a wave function in a fermionic differential operator representation. Within the matrix representation for the fermions, the Lorentz constraint operator includes instead only bosonic momenta as derivatives (a thorough description involving *all* the Lorentz constraints is given in Chap. 7 of Vol. II, implying a vectorial nature for the wave function where only some components of the vector-like wave function are relevant).

Applying the supersymmetry constraints (5.33), (5.34) to Ψ above, we then get from $\mathcal{S}_A\Psi = 0$, $\overline{\mathcal{S}}_A\Psi = 0$ the following system of coupled first order partial differential equations to solve:

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{B} - 2\sqrt{3}a^2\mathsf{B} \,, \tag{5.38}$$

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{A} + 2\sqrt{3}a^2\mathsf{A} = 0 \,. \tag{5.39}$$

In solving these quantum supersymmetry constraints

$$\mathcal{S}_A\Psi = 0 \,, \tag{5.40}$$

$$\overline{\mathcal{S}}_{A'}\Psi = 0 \,, \tag{5.41}$$

from (5.33) and (5.34), the solutions are

$$\Psi = \mathsf{A}_0 \exp\left(-\frac{3a^2}{\hbar}\right) + \mathsf{B}_0 \exp\left(\frac{3a^2}{\hbar}\right)\psi_A\psi^A \,, \tag{5.42}$$

where A_0 and B_0 are independent of a and ψ^A. The exponential factors have a *semi-classical* interpretation as $\exp(-I/\hbar)$, where I is the Euclidean action for a classical solution outside or inside a three-sphere of radius a with a prescribed boundary value of ψ^A. We thus obtain a Hartle–Hawking [18] solution for $\mathsf{B} = 0$ [2, 21, 16, 17].

5.1.3 SUSY (FRW) Quantum States with Cosmological Constant

Following in the footsteps of [32], where the action of $N = 1$ SUGRA with a cosmological constant was introduced, the relevant feature to notice is that the term

$$-\frac{1}{2}e\Upsilon\left(\psi^A_\mu e^\mu{}_{AB'}e^\nu{}_B{}^{B'}\psi^B_\nu\right) + \text{h.c.} \tag{5.43}$$

is added to the action (4.66) of $N = 1$ SUGRA in a 4D spacetime, with $\Lambda \equiv -6\Upsilon^2 < 0$. More precisely,

$$S_\Lambda \equiv -\int \mathrm{d}^4x(\det e)\left[(2\mathsf{k}^2)^{-1}3\Upsilon^2 - \frac{1}{2}\Upsilon(\psi^A{}_\mu e_{AB'}{}^\mu e_B{}^{B'\nu}\psi^B{}_\nu)\right] + \text{h.c.} \tag{5.44}$$

For $\overline{\mathcal{S}}_{A'}\Psi = 0$ and $\mathcal{S}_A\Psi = 0$, we obtain (see [33, 2] for details) the equations

$$\hbar\mathsf{k}^2\frac{\mathrm{d}\mathsf{A}}{\mathrm{d}a} + 48\pi^2 a\mathsf{A} + 18\pi^2\hbar\Upsilon a^2\mathsf{B} = 0\,, \tag{5.45}$$

$$\hbar^2\mathsf{k}^2\frac{\mathrm{d}\mathsf{B}}{\mathrm{d}a} - 48\pi^2\hbar a\mathsf{B} - 256\pi^2\Upsilon a^2 A = 0\,. \tag{5.46}$$

As a consequence, in a supersymmetric FRW model with a cosmological constant term, the coupling between the different fermionic levels *mixes up* the pattern present in the last section. Compare with (5.38) and (5.39). These give second-order equations for $\mathsf{A}(a)$, viz.,

$$\frac{\mathrm{d}^2\mathsf{A}}{\mathrm{d}a^2} - \frac{2}{a}\frac{\mathrm{d}\mathsf{A}}{\mathrm{d}a} - \left(\frac{6}{\hbar} + \frac{36}{\hbar^2}a^2 + \frac{72}{\hbar^2}\Upsilon^2 a^4\right)\mathsf{A} = 0\,, \tag{5.47}$$

which have two independent solutions of the form

$$\mathsf{A} = c_0 + c_2a^2 + c_4a^4 + \cdots\,, \qquad \mathsf{A} = a^3(d_0 + d_2a^2 + d_4a^4 + \cdots)\,, \tag{5.48}$$

convergent for all a. They obey complicated recurrence relations, where c_6 is related to c_4, c_2, and c_0. Hence, regarding a SUSY FRW minisuperspace with a cosmological constant, a quantum bosonic state can be found, namely the Hartle–Hawking solution [18] for an anti-de Sitter case.

In more detail, there are asymptotic solutions of the type

$$\mathsf{A} \sim (\mathsf{B}_0 + \hbar\mathsf{B}_1 + \hbar^2\mathsf{B}_2 + \cdots)\exp(-I/\hbar)\,,$$

with

$$\left(\frac{\mathrm{d}I}{\mathrm{d}a}\right)^2 = 36a^2 72\Upsilon^2 a^4\,, \tag{5.49}$$

from which

$$I = \pm \frac{\pi^2}{\Upsilon^2} \left(1 + 2\Upsilon^2 a^2\right)^{3/2} , \tag{5.50}$$

for $2\Upsilon^2 a^2 < 1$. The minus sign in I corresponds to taking the action of the classical Riemannian solution that smoothly fills the inside of the three-sphere, namely a portion of the four-hyperbola, consistent with $\Lambda < 0$. This gives the Hartle–Hawking state.

5.1.4 Scalar Supermultiplets and Superpotential

The content of the previous sections was both reassuring, since the obtained quantum states were consistent with the set of states in pure bosonic quantum cosmology [34, 35], and encouraging, as the methodology of SQC was introduced through simple and manageable configurations. But the time has come to move towards a more realistic and complete FRW supersymmetric minisuperspace, namely one with a scalar (chiral) supermultiplet formed by the fields $(\phi, \overline{\phi}; \chi_A, \overline{\chi}_{A'})$ and in the presence of a *generic* superpotential, henceforth designated by $\mathsf{P}(\phi, \overline{\phi})$ [15–17, 24]. It is here that the elements introduced in Chap. 3 will be of relevance in providing a broader perspective (see also Sects. 3.3, 3.3.1, and 4.2.3, and also Chap. 3 of Vol. II).

As in Sect. 5.1.1, the FRW supersymmetric minisuperspace is extracted from the action of the more general $N = 1$ SUGRA theory in 4D spacetime, but in the presence of supermatter (see Eq. (25.12) in [36], and also Sect. 3.5), using adequate ansätze for both the bosonic and the fermionic variables. In particular, we employ the ansätze (5.1) and (5.2), where ψ_A, $\overline{\psi}_{A'}$ are time-dependent spinor fields and $\psi_0^A(t)$, $\overline{\psi}_0^{A'}(t)$ are Lagrange multipliers. In addition, a set of time-dependent complex scalar fields $\phi(t)$, $\overline{\phi}(t)$ and their fermionic superpartners, $\chi_A(t)$, $\overline{\chi}_{A'}(t)$ are also included. Finally, a *flat* Kähler manifold is chosen for the complex scalar fields[4]:

$$\mathsf{K} \equiv \phi\overline{\phi} , \tag{5.51}$$

$$\check{\mathrm{D}}_\phi \mathsf{P} \equiv \frac{\partial \mathsf{P}(\phi, \overline{\phi})}{\partial \phi} + \overline{\phi}\mathsf{P} , \tag{5.52}$$

$$\overline{\check{\mathrm{D}}_\phi \mathsf{P}} \equiv \frac{\partial \overline{\mathsf{P}}(\phi, \overline{\phi})}{\partial \overline{\phi}} + \phi\overline{\mathsf{P}} . \tag{5.53}$$

Inserting the above-mentioned ansätze into the general action of $N = 1$ SUGRA with scalar supermatter in a 4D spacetime and integrating over the spatial hypersurfaces, we obtain the extended Lagrangian (inheriting invariance under local time

[4] The reader should note that P will henceforth play the role of W as indicated in Chap. 3 of Vol. II. The corresponding literature in SQC has mainly used P instead (see also [36]).

translation and local Lorentz transformation, and more importantly local $N = 4$ supersymmetry)

$$L_{\rm SUSY}^{\rm FRW} \equiv L_{\rm free} + L_{\rm SPot} \ , \tag{5.54}$$

where $L_{\rm free}$ includes the kinematical and curvature-induced terms, and is now given by

$$\begin{aligned}
L_{\rm free} = & -\frac{3a}{\mathcal{N}}\left[\dot{a} - \frac{\rm i}{4}\left(\overline{\psi}_{0A'}\overline{\psi}^{A'} - \psi^{A}\psi_{0A}\right)\right]^2 \\
& -{\rm i}\frac{3}{2}a^3 n_{AA'}\psi^{A}\overline{\psi}^{A} - {\rm i}\frac{3}{2}a^3 n_{AA'}\overline{\psi}^{A}\dot{\psi}^{A} + 3\mathcal{N}a \\
& +\frac{3}{2}a^2\left(\overline{\psi}_{0A'}\overline{\psi}^{A'} + \psi^{A}\psi_{0A}\right) - \frac{3}{2}\mathcal{N}a^2 n^{AA'}\psi_{A}\overline{\psi}_{A'} \\
& +\frac{3}{16}a^3 n^{AA'}\left(\overline{\psi}_{0A'}\psi_{A}\overline{\psi}^{B'}\overline{\psi}_{B'} + \psi_{0A}\overline{\psi}_{A'}\psi^{B}\psi_{B}\right) \\
& +{\rm i}\frac{1}{4}a^3\left(\phi\dot{\overline{\phi}} - \overline{\phi}\dot{\phi}\right)n^{AA'}\psi_{A}\overline{\psi}_{A'} + \frac{a^3}{\mathcal{N}}\dot{\overline{\phi}}\dot{\phi} \\
& +\frac{\rm i}{2}a^3 n^{AA'}\left(\overline{\chi}_{A'}\dot{\chi}_{A} + \chi_{A}\dot{\overline{\chi}}_{A'}\right) \\
& -\frac{1}{8}a^3 n^{AA'}\left(\overline{\chi}_{A'}\psi_{A}\psi_{0}^{B}\chi_{B} + \overline{\chi}_{A'}\psi_{0A}\psi_{B}\chi^{B}\right) \\
& -\frac{1}{8}a^3 n^{AA'}\left(\chi_{A}\overline{\psi}_{A'}\overline{\psi}_{0}^{B'}\overline{\chi}_{B} + \chi_{A}\overline{\psi}_{0A'}\overline{\psi}_{B'}\overline{\chi}^{B'}\right) \\
& +\frac{\rm i}{4}a^3\left(\dot{\phi}\overline{\phi} - \overline{\phi}\dot{\phi}\right)n^{AA'}\overline{\chi}_{A'}\chi_{A} + \frac{3}{2}\mathcal{N}a^2\overline{\chi}_{A'}\chi_{A} \\
& +\frac{\mathcal{N}a^3}{4}\left(n^{AA'}n^{BB'}\overline{\chi}_{A'}\psi_{A}\overline{\psi}_{B'}\chi_{B} + \overline{\chi}_{A'}\overline{\psi}^{A'}\psi_{A}\chi^{A}\right) \\
& -\frac{a^3}{2\mathcal{N}}\dot{\overline{\phi}}\left(\chi^{A}\psi_{0A} + 3\mathcal{N}n^{AA'}\chi_{A}\overline{\psi}_{A'}\right) \\
& +\frac{a^3}{2\mathcal{N}}\dot{\phi}\left(\overline{\chi}^{A'}\overline{\psi}_{0A'} + 3\mathcal{N}n^{AA'}\overline{\chi}_{A'}\psi_{A}\right) \\
& +\frac{\mathcal{N}a^3}{4}\overline{\chi}_{A'}\overline{\psi}^{A'}\psi_{A}\chi^{A} + \frac{\mathcal{N}a^3}{2}n^{AA'}n^{BB'}\overline{\psi}_{A'}\psi_{A}\chi_{B}\overline{\chi}_{B'} \\
& +\frac{a^3}{2}n_{AA'}\left(\psi_{0}^{A}\overline{\chi}^{A'}\psi_{B}\chi^{B} + \overline{\psi}_{0}^{A'}\chi^{A}\overline{\psi}_{B'}\overline{\psi}^{B'}\right) \\
& -a^3 n_{BB'}\chi^{B}\overline{\chi}^{B'}\left(\psi_{0}^{A}\psi_{A} + \overline{\psi}_{A'}\overline{\psi}_{0}^{A'}\right) \\
& +\frac{a^3}{4\mathcal{N}}\psi_{0A}\overline{\psi}_{0A'}\chi^{A}\overline{\chi}^{A'} - \mathcal{N}a^3\overline{\psi}_{A'}\overline{\chi}^{A'}\psi_{A}\chi^{A} \\
& -\mathcal{N}a^3 n^{AA'}n^{BB'}\chi_{A}\overline{\psi}_{A'}\psi_{B}\overline{\psi}_{B'} + \frac{5\mathcal{N}a^3}{16}\chi^{A}\chi_{A}\overline{\chi}^{A'}\overline{\chi}_{A'} \ ,
\end{aligned} \tag{5.55}$$

and L_{SPot} corresponds to the new contribution from the superpotential terms, written as

$$\begin{aligned}
L_{\text{SPot}} = & -\frac{3}{2}\psi_0^A a^3 e^{K/2}\overline{P} n_A^{A'}\overline{\psi}_{A'} - \frac{3}{2}a^3 e^{K/2} P n^C_{A'}\psi_C\overline{\psi}_0^{A'} \\
& +3\frac{\mathcal{N}}{4}a^3 e^{K/2}\overline{P}\overline{\psi}_{C'}\overline{\psi}^{C'} + 3\frac{\mathcal{N}}{4}a^3 e^{K/2} P\psi^C\psi_C \\
& +\frac{ia^3}{\sqrt{2}}e^{K/2}\,\check{D}_\phi P\chi^A n_{AA'}\overline{\psi}_0^{A'} + \mathcal{N}a^3 e^{K/2}\frac{3}{2\sqrt{2}}\,\check{D}_\phi P\chi^A\psi_A \\
& +i\frac{ia^3}{\sqrt{2}}e^{K/2}\,\check{D}_{\overline{\phi}}\overline{P}\psi_0^A n_A^{A'}\overline{\chi}_{A'} + \mathcal{N}a^3 e^{K/2}\frac{3}{2\sqrt{2}}\,\check{D}_{\overline{\phi}}\overline{P}\overline{\chi}_{A'}\overline{\psi}^{A'} \\
& +\frac{\mathcal{N}}{2}a^3 e^{K/2}\,\check{D}_\phi D_\phi P\chi^A\chi_A + \frac{\mathcal{N}}{2}a^3 e^{K/2} D_{\overline{\phi}}D_{\overline{\phi}}\overline{P}\overline{\chi}_{A'}\overline{\chi}^{A'} \\
& -\mathcal{N}a^3 e^K\left[\left(\overline{\check{D}_\phi P}\right)\left(\check{D}_\phi P\right) - 3\overline{P}P\right] .
\end{aligned} \tag{5.56}$$

Within the SQC framework, the complete Hamiltonian for this FRW minisuperspace will have the (usual) form

$$H = \mathcal{N}\mathcal{H} + \psi_0^A\mathcal{S}_A + \overline{\mathcal{S}}_{A'}\overline{\psi}_0^{A'} + \mathcal{M}^{AB}\mathcal{J}_{AB} + \overline{\mathcal{M}}^{A'B'}\overline{\mathcal{J}}_{A'B'} ,$$

where ψ_0^A and $\overline{\psi}_0^{A'}$ together with $\mathcal{M}^{AB}$ and $\overline{\mathcal{M}}^{A'B'}$ are again Lagrange multipliers [see (5.10)]. The Hamiltonian constraint is now found to be

$$\mathcal{H} \equiv \mathcal{H}^{\text{FRW}}_{\text{SUSY}} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{SPot}} , \tag{5.57}$$

with

$$\begin{aligned}
\mathcal{H}_{\text{free}} = & -\frac{\pi_a^2}{12a} - 3a + \frac{3}{2}a^2 n^{AA'}\left(\psi_A\overline{\psi}_{A'} + \chi_A\overline{\chi}_{A'}\right) \\
& +\frac{1}{a^3}\pi_{\overline{\phi}}\pi_\phi + \frac{3i}{4}\left(\overline{\phi}\pi_{\overline{\phi}} - \phi\pi_\phi\right)n^{AA'}\psi_A\overline{\psi}_{A'} \\
& +\frac{5i}{4}\left(\overline{\phi}\pi_{\overline{\phi}} - \phi\pi_\phi\right)n^{AA'}\overline{\chi}_{A'}\chi_A \\
& +\frac{3}{2}\pi_\phi n^{AA'}\chi_A\overline{\psi}_{A'} - \frac{3}{2}\pi_{\overline{\phi}} n^{AA'}\overline{\chi}_{A'}\psi_A \\
& +\frac{9}{64}a^3\phi\overline{\phi}\psi_A\psi^A\overline{\psi}^{A'}\overline{\psi}_{A'} + \frac{25a^3}{64}\phi\overline{\phi}\chi_A\chi^A\overline{\chi}^{A'}\overline{\chi}_{A'} \\
& +\frac{15}{8}a^3\phi\overline{\phi}n^{AA'}n^{BB'}\overline{\chi}_{A'}\chi_A\psi_B\overline{\psi}_{B'} \\
& +\frac{i9}{32}a^3\phi\psi_A\psi^A\overline{\chi}_{A'}\overline{\psi}^{A'} - \frac{i9}{32}a^3\overline{\phi\psi}_{A'}\overline{\psi}^{A'}\chi_A\psi^A
\end{aligned}$$

$$
\begin{aligned}
&+3a^3 n^{AA'} n^{BB'} \overline{\chi}^{A'} \psi_A \overline{\psi}_{B'} \chi_B \\
&-\frac{a^3}{2} n^{AA'} n^{BB'} \overline{\psi}_{A'} \psi_A \chi_B \overline{\chi}_{B'} \\
&+\frac{a^3}{2} \overline{\chi}_{A'} \overline{\psi}^{A'} \psi_A \chi^A + \frac{5a^3}{16} \chi_A \chi^A \overline{\chi}^{A'} \overline{\chi}_{A'}
\end{aligned}
\tag{5.58}
$$

and

$$
\begin{aligned}
\mathcal{H}_{\rm SPot} = &-\frac{3}{4} a^3 \mathrm{e}^{\mathsf{K}/2} \overline{\mathsf{P}} \overline{\psi}_{C'} \overline{\psi}^{C'} - \frac{3}{4} a^3 \mathrm{e}^{\mathsf{K}/2}\, \mathsf{P} \psi^C \psi_C \\
&-a^3 \mathrm{e}^{\mathsf{K}/2} \frac{3}{2\sqrt{2}}\, \check{\mathrm{D}}_\phi \mathsf{P} \chi^A \psi_A - a^3 \mathrm{e}^{\mathsf{K}/2} \frac{3}{2\sqrt{2}}\, \check{\mathrm{D}}_{\overline{\phi}} \overline{\mathsf{P}} \overline{\chi}_{A'} \overline{\psi}^{A'} \\
&+\frac{1}{2} a^3 \mathrm{e}^{\mathsf{K}/2}\, \check{\mathrm{D}}_\phi \check{\mathrm{D}}_\phi \mathsf{P} \chi^A \chi_A + \frac{1}{2} a^3 \mathrm{e}^{\mathsf{K}/2}\, \check{\mathrm{D}}_{\overline{\phi}} \check{\mathrm{D}}_{\overline{\phi}} \overline{\mathsf{P}} \overline{\chi}_{A'} \overline{\chi}^{A'} \\
&-a^3 \mathrm{e}^{\mathsf{K}} \left[\left(\overline{\check{\mathrm{D}}_\phi \mathsf{P}} \right) \left(\check{\mathrm{D}}_\phi \mathsf{P} \right) - 3 \overline{\mathsf{P}} \mathsf{P} \right] .
\end{aligned}
\tag{5.59}
$$

The last terms in (5.56) and (5.59) should be compared with the expression for V in Chap. 3 (see also (5.51), (5.52), and (5.53) together with (3.147), (3.148), and (3.151)).

We also have additional second class constraints due to the presence of fermionic variables χ_A. Here we also *re*define the fermionic fields χ_A, in order to simplify the Dirac brackets, following the steps described in [24]. In addition to (5.18), (5.19), and (5.20) (see Sect. 5.1.1), we therefore take

$$
\hat{\chi}_A \equiv \frac{\varsigma a^{3/2}}{2^{1/4}(1+\phi\overline{\phi})} \chi_A \ , \qquad \hat{\overline{\chi}}_{A'} \equiv \frac{\varsigma a^{3/2}}{2^{1/4}(1+\phi\overline{\phi})} \overline{\chi}_{A'} \ , \tag{5.60}
$$

the conjugate momenta being

$$
\pi_{\hat{\chi}_A} = -\mathrm{i} n_{AA'} \hat{\overline{\chi}}^{A'} \ , \qquad \pi_{\hat{\overline{\chi}}_{A'}} = -\mathrm{i} n_{AA'} \hat{\chi}^A \ . \tag{5.61}
$$

This pair forms a set of second class constraints, and the resulting Dirac brackets, leading to

$$
\left[\hat{\chi}_A, \hat{\overline{\chi}}_{A'} \right]_{\rm D} = -\mathrm{i} n_{AA'} \ . \tag{5.62}
$$

In the end, we also have

$$
[a, \pi_a]_{\rm D} = 1 \ , \qquad [\phi, \pi_\phi]_{\rm D} = 1 \ , \qquad [\overline{\phi}, \pi_{\overline{\phi}}]_{\rm D} = 1 \ , \tag{5.63}
$$

and the rest of the brackets are zero. Let us again drop the hat over the variables and describe the theory using only unprimed spinors. To this end, it is helpful to define, in addition to (5.22),

$$\overline{\chi}_A = 2n_A{}^{B'}\overline{\chi}_{B'} \ , \tag{5.64}$$

with which the new Dirac brackets are

$$[\chi_A, \overline{\chi}_B]_{\rm D} = -{\rm i}\varepsilon_{AB} \ , \qquad [\psi_A, \overline{\psi}_B]_{\rm D} = {\rm i}\varepsilon_{AB} \ , \tag{5.65}$$

while the rest of the brackets remain unchanged.

Quantum mechanically, the full set of Dirac brackets are replaced by anticommutators or commutators as follows (bearing in mind that we are using units with $\hbar = 1$):

$$\{\chi_A, \overline{\chi}_B\} = \varepsilon_{AB} \ , \qquad \{\psi_A, \overline{\psi}_B\} = -\varepsilon_{AB} \ , \tag{5.66}$$

$$[a, \pi_a] = [\phi, \pi_\phi] = [\overline{\phi}, \pi_{\overline{\phi}}] = {\rm i} \ . \tag{5.67}$$

In the differential operator representation for *both* bosonic and fermionic conjugate momenta, we take $(\chi_A, \psi_A, a, \phi, \overline{\phi})$ to be the coordinates of the configuration space and $(\overline{\chi}_A, \overline{\psi}_A, \pi_a, \pi_\phi, \pi_{\overline{\phi}})$ to be the momentum operators. Subsequently, we choose

$$\overline{\chi}^A \longrightarrow -\frac{\partial}{\partial \chi^A} \ , \qquad \overline{\psi}_A \longrightarrow \frac{\partial}{\partial \psi^A} \ , \qquad \pi_a \longrightarrow -{\rm i}\frac{\partial}{\partial a} \ , \qquad \pi_\phi \longrightarrow -{\rm i}\frac{\partial}{\partial \phi} \ , \tag{5.68}$$

and similarly for $\pi_{\overline{\phi}}$.

Let us obtain expressions for the supersymmetry and Lorentz constraints within a suitable quantum mechanical representation, analysing the terms involving the coefficients in ψ_0^A, $\overline{\psi}_0^{A'}$, $\mathcal{M}^{AB}$, and $\overline{\mathcal{M}}^{A'B'}$ in the complete Hamiltonian. The Lorentz constraints form a set of first class constraints, together with the Hamiltonian and supersymmetry constraints. These constraints satisfy a corresponding algebra, characterized by using Dirac brackets (rather than Poisson brackets) for this $N = 4$ supersymmetric minisuperspace:

- The Lorentz constraint associated with the field variables ψ_A, $\overline{\psi}_{A'}$, χ_A, and $\overline{\chi}_{A'}$ acquires the form

$$\mathcal{J}_{AB} = \psi_{(A}\overline{\psi}_{B)} - \chi_{(A}\overline{\chi}_{B)} = 0 \ , \tag{5.69}$$

where (5.68) brings in the differential operator

$$\mathcal{J}_{AB} = \psi_{(A}\frac{\partial}{\partial \psi^{B)}} + \chi_{(A}\frac{\partial}{\partial \chi^{B)}} \ . \tag{5.70}$$

- At a canonical quantization level, the supersymmetry constraints have the form

$$
\begin{aligned}
\mathcal{S}_A = & -\mathrm{i}\chi_A \frac{\partial}{\partial\phi} - \frac{1}{2\sqrt{3}} a\psi_A \frac{\partial}{\partial a} - \sqrt{3}a^2\psi_A \\
& + \frac{1}{8\sqrt{3}}\psi^C\psi_C \frac{\partial}{\partial\psi^A} - \frac{\mathrm{i}}{4}\overline{\phi}\chi_A\psi_C \frac{\partial}{\partial\psi^C} \\
& + \frac{3}{4\sqrt{3}}\psi_A\chi^B \frac{\partial}{\partial\chi^B} + \frac{\mathrm{i}}{8}\overline{\phi}\chi^B\chi_B \frac{\partial}{\partial\chi^A} \\
& - \mathrm{i}a^3 \check{\mathsf{D}}_{\overline{\phi}}\overline{\mathsf{P}} \frac{\sqrt{2}}{2} \mathrm{e}^{\mathsf{K}/2} \frac{\partial}{\partial\chi^A} \\
& - \frac{\sqrt{3}}{2} a^3 \mathrm{e}^{\mathsf{K}/2}\overline{\mathsf{P}} \frac{\partial}{\partial\psi^A}
\end{aligned}
\tag{5.71}
$$

and

$$
\begin{aligned}
\overline{\mathcal{S}}_A = & \ \mathrm{i}\frac{\partial}{\partial\chi^A}\frac{\partial}{\partial\overline{\phi}} + \frac{1}{2\sqrt{3}} a \frac{\partial}{\partial a}\frac{\partial}{\partial\psi^A} - \sqrt{3}a^2 \frac{\partial}{\partial\psi^A} \\
& + \frac{1}{8\sqrt{3}}\varepsilon^{BC} \frac{\partial}{\partial\psi^C}\frac{\partial}{\partial\psi^B}\psi_A + \frac{\mathrm{i}}{4}\phi \frac{\partial}{\partial\chi^A}\frac{\partial}{\partial\psi^B}\psi^B \\
& - \frac{3}{4\sqrt{3}} \frac{\partial}{\partial\psi^A}\frac{\partial}{\partial\chi^B}\chi^B + \frac{\mathrm{i}}{8}\phi\varepsilon^{BC} \frac{\partial}{\partial\chi^C}\frac{\partial}{\partial\chi^B}\chi_A \\
& - \mathrm{i}a^3 \check{\mathsf{D}}_{\phi}\mathsf{P}\sqrt{2}\mathrm{e}^{\mathsf{K}/2}\chi_A \\
& + \sqrt{3}a^3\mathrm{e}^{\mathsf{K}/2}\mathsf{P}\psi_A \ .
\end{aligned}
\tag{5.72}
$$

> *Note 5.6* A significant point should (again) be emphasized, as in Sect. 5.1.1: *both* supersymmetry constraints are *linear* with respect to the conjugate momenta of *any* bosonic variable. First-order differential equations will be obtained, suggesting another procedure to retrieve the physical states (different to the use of a Wheeler–DeWitt equation in bosonic quantum cosmology).

The Lorentz constraint (5.70) admits the following wave function as a possible solution (see discussion in Sect. 5.1.2), constituted by a partition in fermionic sectors, each associated with a specific bosonic amplitude (functional of the coordinate variables $a, \phi, \overline{\phi}$):

$$
\begin{aligned}
\Psi_{\mathrm{SUSY}}^{\mathrm{FRW}} = & \ \mathsf{A}(a,\phi,\overline{\phi}) + \mathsf{B}(a,\phi,\overline{\phi})\psi^F\psi_F + \mathrm{i}\mathsf{C}(a,\phi,\overline{\phi})\chi^F\psi_F \\
& + \mathsf{D}(a,\phi,\overline{\phi})\chi^F\chi_F + \mathsf{E}(a,\phi,\overline{\phi})\psi^G\psi_G\chi^F\chi_F \ .
\end{aligned}
\tag{5.73}
$$

Note 5.7 Paying attention once again to the essential features here, the reader should note that the quantum mechanical representation of the constraints by differential operators in the bosonic and fermionic conjugate momentum variables provides a simple solution to the Lorentz constraints in our *specific* FRW minisuperspace scenario. This aspect is (again) a strict consequence of the FRW construction (see (5.55), (5.56), (5.57), (5.58), and (5.59)), leading to the Lagrangian or Hamiltonian where *only spinorial* indices are present. When discussing anisotropic Bianchi models, some care must be taken in dealing with the Lorentz invariance of physical states. We recall that both spatial and spinorial indices are present to construct a wave function in a fermionic differential operator representation.

Finally, applying the supersymmetry constraints (5.71) and (5.72) to Ψ above, we then have to solve the following system of first-order coupled partial differential equations from $\mathcal{S}_A \Psi = 0$ and $\overline{\mathcal{S}}_A \Psi = 0$:

$$\frac{a}{\sqrt{3}} \partial_a \mathsf{B} - 2\sqrt{3} a^2 \mathsf{B} - \partial_{\overline{\phi}} \mathsf{C} - \left(\sqrt{3} a^3 e^{\mathsf{K}/2} \mathsf{P} \right) \mathsf{A} = 0 \,, \tag{5.74}$$

$$\partial_{\overline{\phi}} \mathsf{D} + \frac{1}{4\sqrt{3}} a \partial_a \mathsf{C} - \frac{\sqrt{3}}{2} a^2 C - \left(\frac{1}{\sqrt{2}} a^3 e^{\mathsf{K}/2} \check{\mathsf{D}}_{\phi} \mathsf{P} \right) A = 0 \,, \tag{5.75}$$

$$\partial_{\phi} \mathsf{B} - \frac{1}{4\sqrt{3}} a \partial_a \mathsf{C} - \frac{\sqrt{3}}{2} a^2 \mathsf{C} + \left(\sqrt{2} a^3 \check{\mathsf{D}}_{\overline{\phi}} \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{E} = 0 \,, \tag{5.76}$$

$$\partial_{\phi} \mathsf{C} + \frac{1}{\sqrt{3}} a \partial_a \mathsf{D} + 2\sqrt{3} a^2 \mathsf{D} + \left(2\sqrt{3} a^3 \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{E} = 0 \,, \tag{5.77}$$

$$\partial_{\overline{\phi}} \mathsf{E} - \left(\frac{\sqrt{3}}{4} a^3 \mathsf{P} e^{\mathsf{K}/2} \right) \mathsf{C} - \left(\frac{1}{\sqrt{2}} a^3 e^{\mathsf{K}/2} \check{\mathsf{D}}_{\phi} \mathsf{P} \right) \mathsf{B} = 0 \,, \tag{5.78}$$

$$\frac{a}{\sqrt{3}} \partial_a \mathsf{E} - 2\sqrt{3} a^2 \mathsf{E} + \left(\sqrt{3} a^3 e^{\mathsf{K}/2} \mathsf{P} \right) \mathsf{D} - \left(\frac{1}{\sqrt{2}} a^3 e^{\mathsf{K}/2} \check{\mathsf{D}}_{\phi} \mathsf{P} \right) \mathsf{C} = 0 \,, \tag{5.79}$$

$$\partial_{\phi} \mathsf{A} + \left(\sqrt{2} a^3 \check{\mathsf{D}}_{\overline{\phi}} \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{D} + \left(\frac{\sqrt{3}}{2} a^3 \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{C} = 0 \,, \tag{5.80}$$

$$\frac{a}{\sqrt{3}} \partial_a \mathsf{A} + 2\sqrt{3} a^2 \mathsf{A} - \left(\sqrt{2} a^3 \check{\mathsf{D}}_{\overline{\phi}} \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{C} - \left(2\sqrt{3} a^3 \overline{\mathsf{P}} e^{\mathsf{K}/2} \right) \mathsf{B} = 0 \,. \tag{5.81}$$

Expansion of the Superpotential

Analytical expressions for $\mathsf{A}, \ldots, \mathsf{E}$ solving (5.74), (5.75), (5.76), (5.77), (5.78), (5.79), (5.80), and (5.81) seem difficult to obtain. In the special case $\mathsf{P} = \check{\mathsf{D}}_{\phi} \mathsf{P} = 0$ and their Hermitian conjugates, a particular set of analytical solutions exists [15–17, 24]. This is due to the fact that the equations for A and E are decoupled, and also that the equations for B, C, and D can be separated into solvable second-order differential equations pointing to, e.g., wormhole (Hawking–Page) states [15]. When P is a constant ($\phi \to \phi_0$) and there are (consistently) no associated fermions $\chi_A(t)$, $\overline{\chi}_A(t)$, we obtain an effective cosmological constant. Solutions for this case were partially obtained in [33, 37] (see Sect. 5.1.3). However, the presence of a generic P or $\check{\mathsf{D}}_\phi \mathsf{P}$ imply that the fermionic sectors in Ψ will be severely coupled and not easily separated.

We will henceforth employ an asymptotic formulation of possible solutions of (5.74), (5.75), (5.76), (5.77), (5.78), (5.79), (5.80), and (5.81). More precisely, we introduce a *perturbative* expansion[5] for $\mathsf{A}, \dots, \mathsf{E}$ and the potential P as follows [38, 24]:

$$\mathsf{A} = \sum \ell^i \mathsf{A}_i = \ell^0 \mathsf{A}_0 + \ell^1 \mathsf{A}_1 + \ell^2 \mathsf{A}_2 + \cdots \ , \quad i = 0, 1, 2, \dots \ , \tag{5.82}$$

$$\mathsf{P} = \sum \ell^n \mathsf{P}^{(n)} = \ell^0 \mathsf{P}^{(0)} + \ell^1 \mathsf{P}^{(1)} + \ell^2 \mathsf{P}^{(2)} + \cdots \ , \quad n = 0, 1, 2, \dots \ , \tag{5.83}$$

with $\mathsf{P}^{(0)} = 0$ and $\ell \ll 1$. As a consequence, from each of the equations for A to E in (5.74), (5.75), (5.76), (5.77), (5.78), (5.79), (5.80), and (5.81), we obtain up to first order (choosing $\mathsf{K} = \overline{\phi}\phi$), the following two equations for orders ℓ^0 and ℓ^1, respectively:

$$\ell^0 \longrightarrow \partial_{\overline{\phi}} \mathsf{E}_0 = 0 \ , \tag{5.84}$$

$$\ell^1 \longrightarrow \partial_{\overline{\phi}} \mathsf{E}_1 - \left[\frac{\sqrt{3}}{4} a^3 \mathsf{P}^{(1)} \mathrm{e}^{\phi\overline{\phi}/2} \right] \mathsf{C}_0 - \left[\frac{1}{\sqrt{2}} a^3 \mathrm{e}^{\phi\overline{\phi}/2} \check{\mathsf{D}}_\phi \mathsf{P}^{(1)} \right] \mathsf{B}_0 = 0 \ , \tag{5.85}$$

$$\ell^0 \longrightarrow \frac{a}{\sqrt{3}} \partial_a \mathsf{E}_0 - 2\sqrt{3} a^2 \mathsf{E}_0 = 0 \ , \tag{5.86}$$

$$\begin{aligned} \ell^1 \longrightarrow & \frac{a}{\sqrt{3}} \partial_a \mathsf{E}_1 - 2\sqrt{3} a^2 \mathsf{E}_1 + \left[\sqrt{3} a^3 \mathrm{e}^{\phi\overline{\phi}/2} \mathsf{P}^{(1)} \right] \mathsf{D}_0 \\ & - \left[\frac{1}{\sqrt{2}} a^3 \mathrm{e}^{\phi\overline{\phi}/2} \check{\mathsf{D}}_\phi \mathsf{P}^{(1)} \right] \mathsf{C}_0 = 0 \ , \end{aligned} \tag{5.87}$$

$$\ell^0 \longrightarrow \frac{a}{\sqrt{3}} \partial_a \mathsf{B}_0 - 2\sqrt{3} a^2 \mathsf{B}_0 - \sqrt{3} \mathsf{B}_0 - \partial_{\overline{\phi}} \mathsf{C}_0 - \frac{1}{2} \phi \mathsf{C}_0 = 0 \ , \tag{5.88}$$

$$\begin{aligned} \ell^1 \longrightarrow & \frac{a}{\sqrt{3}} \partial_a \mathsf{B}_1 - 2\sqrt{3} a^2 \mathsf{B}_1 - \sqrt{3} \mathsf{B}_1 - \partial_{\overline{\phi}} \mathsf{C}_1 \\ & - \frac{1}{2} \phi \mathsf{C}_1 + \left[\sqrt{3} a^3 \mathrm{e}^{\phi\overline{\phi}/2} \mathsf{P}^{(1)} \right] \mathsf{A}_0 = 0 \ , \end{aligned} \tag{5.89}$$

---

[5] Please notice that we are *not* introducing a Fourier-like expansion in the variables $a, \phi, \overline{\phi}$. This would lead to a midisuperspace scenario [39, 40], a setting situated somewhere between a SQC minisuperspace and the full theory of supergravity. Instead, we restrict ourselves to a strongly truncated supersymmetric minisuperspace and employ the expansion as a simplifying assumption in our equations.

$$\ell^0 \longrightarrow \partial_{\overline{\phi}}\mathsf{D}_0 + \frac{1}{2}\phi\mathsf{D}_0 + \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}_0 - \frac{\sqrt{3}}{2}a^2\mathsf{C}_0 - \frac{1}{4\sqrt{3}}\mathsf{C}_0 = 0\,, \tag{5.90}$$

$$\ell^1 \longrightarrow \partial_{\overline{\phi}}\mathsf{D}_1 + \frac{1}{2}\phi\mathsf{D}_1 + \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}_1 - \frac{\sqrt{3}}{2}a^2\mathsf{C}_1 - \frac{1}{4\sqrt{3}}\mathsf{C}_1 - \left[\frac{1}{\sqrt{2}}a^3\mathrm{e}^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi P^{(1)}\right]\mathsf{A}_0 = 0\,, \tag{5.91}$$

$$\ell^0 \longrightarrow \partial_\phi\mathsf{A}_0 = 0\,, \tag{5.92}$$

$$\ell^1 \longrightarrow \partial_\phi\mathsf{A}_1 + \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\mathrm{e}^{\phi\overline{\phi}/2}\right]\mathsf{D}_0 + \left[\frac{\sqrt{3}}{2}a^3\overline{\mathsf{P}}^{(1)}\mathrm{e}^{\phi\overline{\phi}/2}\right]\mathsf{C}_0 = 0\,, \tag{5.93}$$

$$\ell^0 \longrightarrow \frac{a}{\sqrt{3}}\partial_a\mathsf{A}_0 + 2\sqrt{3}a^2\mathsf{A}_0 = 0\,, \tag{5.94}$$

$$\ell^1 \longrightarrow \frac{a}{\sqrt{3}}\partial_a\mathsf{A}_1 + 2\sqrt{3}a^2\mathsf{A}_1 - \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\mathrm{e}^{\phi\overline{\phi}/2}\right]\mathsf{C}_0 + \left[2\sqrt{3}a^3\overline{\mathsf{P}}^{(1)}\mathrm{e}^{\phi\overline{\phi}/2}\right]\mathsf{B}_0 = 0\,, \tag{5.95}$$

$$\ell^0 \longrightarrow \frac{a}{\sqrt{3}}\partial_a\mathsf{D}_0 + 2\sqrt{3}a^2\mathsf{D}_0 - \sqrt{3}\mathsf{D}_0 + \partial_\phi\mathsf{C}_0 + \frac{1}{2}\overline{\phi}\mathsf{C}_0 = 0\,, \tag{5.96}$$

$$\ell^1 \longrightarrow \frac{a}{\sqrt{3}}\partial_a\mathsf{D}_1 + 2\sqrt{3}a^2\mathsf{D}_1 - \sqrt{3}\mathsf{D}_1 + \partial_\phi\mathsf{C}_0 + \frac{1}{2}\overline{\phi}\mathsf{C}_0 + \left[2\sqrt{3}a^3\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\right]\mathsf{E}_0 = 0\,, \tag{5.97}$$

$$\ell^0 \longrightarrow \partial_\phi\mathsf{B}_0 + \frac{1}{2}\overline{\phi}\mathsf{B}_0 - \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}_0 - \frac{\sqrt{3}}{2}a^2\mathsf{C}_0 + \frac{1}{4\sqrt{3}}\mathsf{C}_0 = 0\,, \tag{5.98}$$

$$\ell^1 \longrightarrow \partial_\phi\mathsf{B}_1 + \frac{1}{2}\overline{\phi}\mathsf{B}_1 - \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}_1 - \frac{\sqrt{3}}{2}a^2\mathsf{C}_1 + \frac{1}{4\sqrt{3}}\mathsf{C}_1 + \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\mathrm{e}^{\phi\overline{\phi}/2}\right]\mathsf{E}_0 = 0\,. \tag{5.99}$$

Redefining $\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{E}_0, \mathsf{E}_1$ by

$$\mathsf{A}_0(a,\phi,\overline{\phi}) = \mathsf{A}_0'(a,\phi,\overline{\phi})\mathrm{e}^{-\phi\overline{\phi}/2}\,, \tag{5.100}$$

we further retrieve the following equations at order ℓ^0:

$$\partial_{\overline{\phi}}\mathsf{E}_0' - \frac{1}{2}\phi\mathsf{E}_0' = 0\,, \tag{5.101}$$

$$\frac{a}{\sqrt{3}}\partial_a \mathsf{E}'_0 - 2\sqrt{3}a^2\mathsf{E}'_0 = 0\,, \tag{5.102}$$

$$\frac{a}{\sqrt{3}}\partial_a \mathsf{B}'_0 - 2\sqrt{3}a^2\mathsf{B}'_0 - \sqrt{3}\mathsf{B}'_0 - \partial_{\overline{\phi}}\mathsf{C}'_0 = 0\,, \tag{5.103}$$

$$\partial_{\overline{\phi}}\mathsf{D}'_0 + \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}'_0 - \frac{\sqrt{3}}{2}a^2\mathsf{C}'_0 - \frac{1}{4\sqrt{3}}\mathsf{C}'_0 = 0\,, \tag{5.104}$$

$$\partial_\phi \mathsf{A}'_0 - \frac{1}{2}\overline{\phi}\mathsf{A}'_0 = 0\,, \tag{5.105}$$

$$\frac{a}{\sqrt{3}}\partial_a \mathsf{A}'_0 + 2\sqrt{3}a^2\mathsf{A}'_0 = 0\,, \tag{5.106}$$

$$\frac{a}{\sqrt{3}}\partial_a \mathsf{D}'_0 + 2\sqrt{3}a^2\mathsf{D}'_0 - \sqrt{3}\mathsf{D}'_0 + \partial_\phi\mathsf{C}'_0 = 0\,, \tag{5.107}$$

$$\partial_\phi\mathsf{B}'_0 - \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}'_0 - \frac{\sqrt{3}}{2}a^2\mathsf{C}'_0 + \frac{1}{4\sqrt{3}}\mathsf{C}'_0 = 0\,, \tag{5.108}$$

whereas at order ℓ^1 :

$$\partial_{\overline{\phi}}\mathsf{E}'_1 - \frac{1}{2}\phi\mathsf{E}'_1 - \left[\frac{\sqrt{3}}{4}a^3\mathsf{P}^{(1)}e^{\phi\overline{\phi}/2}\right]\mathsf{C}'_0 - \left[\frac{1}{\sqrt{2}}a^3e^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}\right]\mathsf{B}'_0 = 0\,, \tag{5.109}$$

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{E}'_1 - 2\sqrt{3}a^2\mathsf{E}'_1 + \left[\sqrt{3}a^3e^{\phi\overline{\phi}/2}\mathsf{P}^{(1)}\right]\mathsf{D}'_0 - \left[\frac{1}{\sqrt{2}}a^3e^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}\right]\mathsf{C}'_0 = 0\,, \tag{5.110}$$

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{B}'_1 - 2\sqrt{3}a^2\mathsf{B}'_1 - \sqrt{3}\mathsf{B}'_1 - \partial_{\overline{\phi}}\mathsf{C}'_1 + \left[\sqrt{3}a^3e^{\phi\overline{\phi}/2}\mathsf{P}^{(1)}\right]\mathsf{A}'_0 = 0\,, \tag{5.111}$$

$$\partial_{\overline{\phi}}\mathsf{D}'_1 + \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}'_1 - \frac{\sqrt{3}}{2}a^2\mathsf{C}'_1 - \frac{1}{4\sqrt{3}}\mathsf{C}'_1 - \left[\frac{1}{\sqrt{2}}a^3e^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}\right]\mathsf{A}'_0 = 0\,, \tag{5.112}$$

$$\partial_\phi\mathsf{A}'_1 - \frac{1}{2}\overline{\phi}\mathsf{A}'_1 + \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}e^{\phi\overline{\phi}/2}\right]\mathsf{D}'_0 + \left[\frac{\sqrt{3}}{2}a^3\overline{\mathsf{P}}^{(1)}e^{\phi\overline{\phi}/2}\right]\mathsf{C}'_0 = 0\,, \tag{5.113}$$

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{A}'_1 + 2\sqrt{3}a^2\mathsf{A}'_1 - \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}e^{\phi\overline{\phi}/2}\right]\mathsf{C}'_0 + \left[2\sqrt{3}a^3\overline{\mathsf{P}}^{(1)}e^{\phi\overline{\phi}/2}\right]\mathsf{B}'_0 = 0\,, \tag{5.114}$$

$$\frac{a}{\sqrt{3}}\partial_a\mathsf{D}'_1 + 2\sqrt{3}a^2\mathsf{D}'_1 - \sqrt{3}\mathsf{D}'_1 + \partial_\phi\mathsf{C}'_1 + \left[2\sqrt{3}a^3e^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\right]\mathsf{E}'_0 = 0, \tag{5.115}$$

$$\partial_\phi\mathsf{B}'_1 - \frac{1}{4\sqrt{3}}a\partial_a\mathsf{C}'_1 - \frac{\sqrt{3}}{2}a^2\mathsf{C}'_1 + \frac{1}{4\sqrt{3}}\mathsf{C}'_1 + \left[\sqrt{2}a^3\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}e^{\phi\overline{\phi}/2}\right]\mathsf{E}'_0 = 0\,. \tag{5.116}$$

From (5.103), (5.104), (5.107), and (5.108), we get the following equations for B_0', C_0', and D_0':

$$\frac{a^2}{12}\frac{\partial^2 \mathsf{C}_0'}{\partial a^2} - \frac{\partial^2 \mathsf{C}_0'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{C}_0'}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}_0' = 0\,, \tag{5.117}$$

$$\frac{a^2}{12}\frac{\partial^2 \mathsf{B}_0'}{\partial a^2} - \frac{\partial^2 \mathsf{B}_0'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{B}_0'}{\partial a} + \left(-3a^4 + \frac{1}{4} - 2a^2\right)\mathsf{B}_0' = 0\,, \tag{5.118}$$

$$\frac{a^2}{12}\frac{\partial^2 \mathsf{D}_0'}{\partial a^2} - \frac{\partial^2 \mathsf{D}_0'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{D}_0'}{\partial a} + \left(-3a^4 + \frac{1}{4} + 2a^2\right)\mathsf{D}_0' = 0\,. \tag{5.119}$$

To be more precise, we obtain two equations equal to (5.117), which confirms the consistency of the approach at order ℓ^0. At order ℓ^1, the same procedure leads to the following equations from (5.111), (5.112), (5.115), and (5.116):

$$\begin{aligned}0 = {} & \frac{a^2}{12}\frac{\partial^2 \mathsf{B}_1'}{\partial a^2} - \frac{\partial^2 \mathsf{B}_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{B}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4} - 2a^2\right)\mathsf{B}_1' \\ & + \frac{1}{4}\mathrm{e}^{\phi\overline{\phi}/2}\mathsf{P}^{(1)} a\partial_a\left(a^3\mathsf{A}_0'\right) + \frac{3}{2}a^5\mathrm{e}^{\phi\overline{\phi}/2}\mathsf{P}^{(1)}\mathsf{A}_0' \\ & - \frac{1}{4}a^3\mathrm{e}^{\phi\overline{\phi}/2}\mathsf{P}^{(1)}\mathsf{A}_0' - \sqrt{2}a^3\partial_{\overline{\phi}}\left[\mathrm{e}^{\phi\overline{\phi}/2}\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\right]\mathsf{E}_0'\end{aligned} \tag{5.120}$$

and

$$\begin{aligned}0 = {} & \frac{a^2}{12}\frac{\partial^2 \mathsf{D}_1'}{\partial a^2} - \frac{\partial^2 \mathsf{D}_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{D}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4} + 2a^2\right)\mathsf{D}_1' \\ & + \frac{1}{2}\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)} a\partial_a\left(a^3\mathsf{E}_0'\right) - 3a^5\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\mathsf{E}_0' \\ & - \frac{1}{2}a^3\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\mathsf{E}_0' + \frac{1}{\sqrt{2}}a^3\partial_{\overline{\phi}}\left[\mathrm{e}^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}\right]\mathsf{A}_0'\,.\end{aligned} \tag{5.121}$$

However, a significant difference occurs in contrast to (5.117). At order ℓ^1, the presence of $\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}$, $\mathsf{P}^{(1)}$, and their Hermitian conjugates, together with A_0' and E_0', determines two *inequivalent* equations for C_1' :

$$\begin{aligned}0 = {} & \frac{a^2}{12}\frac{\partial^2 \mathsf{C}_1'}{\partial a^2} - \frac{\partial^2 C_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{C}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}_1' \\ & - \sqrt{\frac{2}{3}}\mathrm{e}^{\phi\overline{\phi}/2}\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}} a\partial_a\left(a^3\mathsf{E}_0'\right) + 2\sqrt{6}a^5\mathrm{e}^{\phi\overline{\phi}/2}\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\mathsf{E}_0' \\ & - \sqrt{6}a^3\mathrm{e}^{\phi\overline{\phi}/2}\overline{\check{\mathsf{D}}_\phi\mathsf{P}^{(1)}}\mathsf{E}_0' + \sqrt{3}a^3\partial_\phi\left[\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\mathsf{A}_0'\right]\end{aligned} \tag{5.122}$$

and

$$0 = \frac{a^2}{12}\frac{\partial^2 \mathsf{C}_1'}{\partial a^2} - \frac{\partial^2 \mathsf{C}_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{C}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}_1'$$
$$-\frac{a}{\sqrt{6}}\mathrm{e}^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi \mathsf{P}^{(1)} a\partial_a\left(a^3\mathsf{A}_0'\right) - \sqrt{6}a^5\mathrm{e}^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi \mathsf{P}^{(1)}\mathsf{A}_0'$$
$$+\frac{\sqrt{3}}{\sqrt{2}}a^3\mathrm{e}^{\phi\overline{\phi}/2}\check{\mathsf{D}}_\phi \mathsf{P}^{(1)}\mathsf{A}_0' - 2\sqrt{3}a^3\partial_{\overline{\phi}}\left[\mathrm{e}^{\phi\overline{\phi}/2}\overline{\mathsf{P}}^{(1)}\mathsf{E}_0'\right] . \tag{5.123}$$

Using equations for A_0 and E_0 at order ℓ^0, the above equations can be further simplified to

$$0 = \frac{a^2}{12}\frac{\partial^2 \mathsf{C}_1'}{\partial a^2} - \frac{\partial^2 \mathsf{C}_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{C}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}_1' + \sqrt{3}a^3\check{\mathsf{D}}_\phi \mathsf{P}^{(1)}\mathrm{e}^{\phi\overline{\phi}/2}\mathsf{A}_0' , \tag{5.124}$$

$$0 = \frac{a^2}{12}\frac{\partial^2 \mathsf{C}_1'}{\partial a^2} - \frac{\partial^2 \mathsf{C}_1'}{\partial\phi\partial\overline{\phi}} - \frac{a}{4}\frac{\partial \mathsf{C}_1'}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}_1' - 2\sqrt{3}a^3\overline{\check{\mathsf{D}}_\phi \mathsf{P}^{(1)}}\mathrm{e}^{\phi\overline{\phi}/2}\mathsf{E}_0' . \tag{5.125}$$

Solutions of the Supersymmetry Constraints

Class of Solutions at Order ℓ^0

Let us now discuss how to solve (5.101), (5.102), (5.105), and (5.106) explicitly [24]. They are decoupled and constitute linear first order partial differential equations. Using also (5.100), we thus obtain

$$\mathsf{A}_0 = c_1 r^{\varsigma_1}\mathrm{e}^{-\mathrm{i}\varsigma_1\theta}\mathrm{e}^{-3a^2} = d_1\mathrm{e}^{k_1(\phi_1-\mathrm{i}\phi_2)}\mathrm{e}^{-3a^2} = f(\overline{\phi})\mathrm{e}^{-3a^2} , \tag{5.126}$$
$$\mathsf{E}_0 = c_2 r^{\varsigma_2}\mathrm{e}^{\mathrm{i}\varsigma_2\theta}\mathrm{e}^{3a^2} = d_2\mathrm{e}^{k_2(\phi_1+\mathrm{i}\phi_2)}\mathrm{e}^{3a^2} = g(\phi)\mathrm{e}^{3a^2} , \tag{5.127}$$

with

$$\phi = r\mathrm{e}^{i\theta} = \phi_1 + \mathrm{i}\phi_2 , \tag{5.128}$$

where $c_1, c_2, \varsigma_1, \varsigma_2, d_1, d_2, k_1, k_2$ are constants and f, g are anti-holomorphic and holomorphic functions of ϕ and $\overline{\phi}$, respectively.

Now consider (5.117), (5.118), and (5.119) for the functionals B_0', C_0', and D_0', where (5.128) will allow us to write the expressions for B_0, C_0, and D_0. In [15–17], the supersymmetry constraints were solved with $\mathsf{P} = \check{\mathsf{D}}_\phi\mathsf{P} = 0$. The ground state wormhole was obtained as a wave function in the form $\Psi = \mathsf{B}\psi^2 + \mathsf{C}\psi\chi + \mathsf{D}\chi^2$, complying with the necessary boundary conditions. Basically, these were introduced by solving the corresponding Euclidean Hamilton–Jacobi equation and writing the solutions of the effective Wheeler–DeWitt equation as $\mathrm{e}^{\pm I}$ by means of a power series. Moreover, some specific symmetry assumptions directly related to the geometrical (e.g., wormhole) configuration were imposed, which simplified the corresponding partial differential equations and hence the solutions.

Note 5.8 As far as the Hartle–Hawking solution is concerned, most of the SQC solutions found in the literature bear *some* of its properties. In particular, for supermultiplets, the form of the supersymmetry constraints that were used could not determine the complete dependence of Ψ on the scalar field [20]. Some improvements were made in [4, 41, 23, 24]. By employing $\phi = r\mathrm{e}^{\mathrm{i}\theta} = \phi_1 + \mathrm{i}\phi_2$, we were able to effectively decouple the two degrees of freedom associated with the complex scalar fields, whereupon the supersymmetry constraints became more manageable. A pleasant consequence of this approach was to provide a sensible framework for discussing whether conserved currents can be defined in supersymmetric quantum cosmology [4, 41, 23].

We now obtain the *generic* set of solutions by focusing initially on (5.117) *without* any symmetry restrictions. Again we will use (5.128). We can thus write

$$\frac{a^2}{12}\frac{\partial^2 \mathsf{C}'_0}{\partial a^2} - \left(\frac{\partial^2 \mathsf{C}'_0}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{C}'_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathsf{C}'_0}{\partial \theta^2}\right) - \frac{a}{4}\frac{\partial \mathsf{C}'_0}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\mathsf{C}'_0 = 0 \; . \tag{5.129}$$

Redefining

$$\mathsf{C}'_0 \equiv \xi(a) Q(r,\theta) \; , \tag{5.130}$$

we get the following equations

$$\frac{a^2}{12}\frac{\partial^2 \xi}{\partial a^2} - \frac{a}{4}\frac{\partial \xi}{\partial a} + \left(-3a^4 + \frac{1}{4}\right)\xi = \xi\varepsilon \; , \tag{5.131}$$

$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Q}{\partial \theta^2} = -\varepsilon Q \; , \tag{5.132}$$

where ε is a separation constant. We further employ

$$\xi(a) \equiv a\Gamma(a) \; , \tag{5.133}$$

$$Q \equiv R(r)\Theta(\theta) \; . \tag{5.134}$$

From (5.131), we obtain

$$a^2\frac{\partial^2 \Gamma}{\partial a^2} - a\frac{\partial \Gamma}{\partial a} + \left(-36a^4 - 12\varepsilon\right)\Gamma = 0 \; , \tag{5.135}$$

and (5.132) leads to

$$\frac{\partial^2 \Theta}{\partial \theta^2} - \delta\Theta = 0 , \tag{5.136}$$

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + R\left(r^2\varepsilon + \delta\right) = 0 , \tag{5.137}$$

where δ is another constant. Solutions for (5.135) can be expressed in the form of Bessel functions Z_ν by

$$\Gamma(a) = a^p Z_\nu(\lambda a^q) , \qquad z \equiv \gamma a^q , \tag{5.138}$$

with $p = 1, q = 2, \gamma \equiv \mathrm{i}/2, \upsilon^2 \equiv (12\varepsilon + p^2)/q^2$. As far as (5.136) and (5.137) are concerned, we obtain, respectively,

$$R(r) = r^p Z_\nu(\lambda r^q) , \tag{5.139}$$

$$\Theta(\theta) = \begin{cases} \mathrm{e}^{\pm\theta\sqrt{\delta}} , & \delta > 0 , \\ \mathrm{e}^{\pm\mathrm{i}\theta\sqrt{|\delta|}} , & \delta < 0 , \end{cases} \tag{5.140}$$

with $p = 0, q = 1, \lambda^2 = \varepsilon, \upsilon^2 = (p^2 - \varepsilon)/q^2$. The Bessel functions can be of first, second and Hankel type, viz., $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, and $H_\nu^{(2)}(z)$, respectively. Their asymptotic behavior for small values of the argument is ($z \equiv \gamma a^q$)

$$J_\nu(z) \sim \left(\frac{1}{2}z\right)^\nu , \tag{5.141}$$

$$Y_\nu(z) \sim -\mathrm{i}H_\nu^{(1)}(z) \sim \mathrm{i}H_\nu^{(2)}(z) \sim -\frac{1}{\pi}\left(\frac{1}{2}z\right)^\nu , \tag{5.142}$$

and for large values of the argument ($z \equiv \gamma a^q$),

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4\pi}\right) , \tag{5.143}$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4\pi}\right) , \tag{5.144}$$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left[\mathrm{i}\left(z - \frac{1}{2}\nu\pi - \frac{1}{4\pi}\right)\right] , \tag{5.145}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left[-\mathrm{i}\left(z - \frac{1}{2}\nu\pi - \frac{1}{4\pi}\right)\right] . \tag{5.146}$$

Regarding the equations for B_0' and D_0', the same method for (5.118) and (5.119) leads to the same formal dependence on ϕ and $\overline{\phi}$ or r and θ. For the scale factor dependence, we get instead

$$a^2 \frac{\partial^2 \Gamma}{\partial a^2} - a \frac{\partial \Gamma}{\partial a} + \left(-36a^4 \pm 24a^2 - 12\varepsilon\right) \Gamma = 0 \,. \tag{5.147}$$

Multiplying by $1/a^2$ and using

$$\Gamma(a) \equiv g(a) \exp\left(\frac{1}{2} \int \frac{1}{b} \mathrm{d}b\right) ,$$

we find that $g(a)$ satisfies the equation

$$\frac{\partial^2 g}{\partial a^2} + \varphi(a) g = 0 \,, \qquad \varphi(a) \equiv \frac{1}{a^2} \left(-36a^4 \pm 24a^2 - 12\varepsilon\right) - \frac{3}{4a^2} \,, \tag{5.148}$$

whose solutions are asymptotically Airy functions $Ai(z)$, $Bi(z)$, with $z \equiv a$:

$$Ai(z) = \pi^{-1} \sqrt{\frac{z}{3}} K_{1/3} \left(\frac{2}{3} z^{3/2}\right) , \tag{5.149}$$

$$Bi(z) = \sqrt{\frac{z}{3}} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2}\right) + I_{1/3} \left(\frac{2}{3} z^{3/2}\right)\right] , \tag{5.150}$$

where K and I are modified Bessel functions, which behave for large values of the argument as

$$Ai \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \sum_{k=0}^{\infty} (-1)^k c_k \left(\frac{2}{3} z^{3/2}\right)^{-k} , \tag{5.151}$$

$$Bi \sim \pi^{-1/2} z^{-1/4} \exp\left(\frac{2}{3} z^{3/2}\right) \sum_{k=0}^{\infty} c_k \left(\frac{2}{3} z^{3/2}\right)^{-k} . \tag{5.152}$$

If we take the limit of large a and neglect the a^2 term in the potential of (5.147), we get the same approximate expression for large argument dependence, similarly to the case of B_0'. As an alternative, we can also write expressions for B_0' and D_0' by solving (5.118) and (5.119), using instead (5.103), (5.107), (5.104), (5.108) and the expression (possibly approximate) for C_0'. The choice of boundary conditions, determined by the physical configuration under investigation, e.g., wormhole (Hawking–Page) [19] or no-boundary (Hartle–Hawking) [18] state, establishes which type of Bessel function will be selected. In particular, note that, from the analysis of C_0', the choice of $H_\nu^{(1)}(z)$ or $H_\nu^{(2)}(z)$ leads to $\mathrm{e}^{-\gamma a^2}$ or $\mathrm{e}^{\gamma a^2}$ in the large scale factor limit, with γ a constant. These would constitute the Hawking–Page dominance for a wormhole state or the Hartle–Hawking behavior for a closed FRW model.

Class of Solutions at Order ℓ^1

At order ℓ^1 and making the choices

$$\check{\mathsf{D}}_\phi \mathsf{P}^{(1)} = \overline{\check{\mathsf{D}}_\phi \mathsf{P}^{(1)}} = 0 \, , \tag{5.153}$$

we can then write [24]

$$\partial_{\overline{\phi}} \mathsf{E}'_1 - \frac{1}{2}\phi \mathsf{E}'_1 - \frac{\sqrt{3}}{4} a^3 \mathsf{P}^{(1)} e^{\phi\overline{\phi}/2} \mathsf{C}'_0 = 0 \, , \tag{5.154}$$

$$\frac{a}{\sqrt{3}} \partial_a \mathsf{E}'_1 - 2\sqrt{3} a^2 \mathsf{E}'_1 + \sqrt{3} a^3 e^{\phi\overline{\phi}/2} \mathsf{P}^{(1)} \mathsf{D}'_0 = 0 \, , \tag{5.155}$$

$$\partial_\phi \mathsf{A}'_1 - \frac{1}{2}\overline{\phi} \mathsf{A}'_1 + \frac{\sqrt{3}}{2} a^3 \overline{\mathsf{P}}^{(1)} e^{\phi\overline{\phi}/2} \mathsf{C}'_0 = 0 \, , \tag{5.156}$$

$$\frac{a}{\sqrt{3}} \partial_a \mathsf{A}'_1 + 2\sqrt{3} a^2 \mathsf{A}'_1 + 2\sqrt{3} a^3 \overline{\mathsf{P}}^{(1)} e^{\phi\overline{\phi}/2} \mathsf{B}'_0 = 0 \, , \tag{5.157}$$

and using (5.102) and (5.106) to simplify (5.120) and (5.121), we also obtain

$$0 = \frac{a^2}{12} \frac{\partial^2 \mathsf{B}'_1}{\partial a^2} - \frac{\partial^2 \mathsf{B}'_1}{\partial \phi \partial \overline{\phi}} - \frac{a}{4} \frac{\partial \mathsf{B}'_1}{\partial a} + \left(-3a^4 + \frac{1}{4} - 2a^2 \right) \mathsf{B}'_1 + \frac{1}{2} e^{\phi\overline{\phi}/2} \mathsf{P}^{(1)} \mathsf{A}'_0 \, , \tag{5.158}$$

$$0 = \frac{a^2}{12} \frac{\partial^2 D'_1}{\partial a^2} - \frac{\partial^2 D'_1}{\partial \phi \partial \overline{\phi}} - \frac{a}{4} \frac{\partial D'_1}{\partial a} + \left(-3a^4 + \frac{1}{4} + 2a^2 \right) D'_1 + e^{\phi\overline{\phi}/2} \mathsf{P}^{(1)} E'_0 \, , \tag{5.159}$$

$$0 = \frac{a^2}{12} \frac{\partial^2 \mathsf{C}'_1}{\partial a^2} - \frac{\partial^2 \mathsf{C}'_1}{\partial \phi \partial \overline{\phi}} - \frac{a}{4} \frac{\partial \mathsf{C}'_1}{\partial a} + \left(-3a^4 + \frac{1}{4} \right) \mathsf{C}'_1 \, . \tag{5.160}$$

The solution of the latter equation follows basically from the method employed for C'_0, as previously described. This seems to indicate that, as far as the intermediate fermion sector $\psi\chi$ of Ψ in (5.37) is concerned, the corresponding bosonic functional will be unaffected by terms in P and $\check{\mathsf{D}}_\phi \mathsf{P}$. Hence, we would recover for C a dependence in a, ϕ, and $\overline{\phi}$ of the type hereby established.

Regarding A'_1 and E'_1, the situation is as follows. Equations (5.155) and (5.157) can be integrated to obtain

$$\mathsf{A}'_1 \sim e^{-\gamma_1 a^2} \left[\Omega(r,\theta) \int e^{\gamma_1 a^2} W(a) + \beta \right] \, , \tag{5.161}$$

$$\mathsf{E}'_1 \sim e^{\gamma_2 a^2} \left[\tilde{\Omega}(r,\theta) \int e^{-\gamma_2 a^2} \tilde{W}(a) + \tilde{\beta} \right] \, , \tag{5.162}$$

where γ_1, γ_2, β, and $\tilde{\beta}$ are constants, and $\Omega(r,\theta)$, $\tilde{\Omega}(r,\theta)$, $W(a)$, and $\tilde{W}(a)$ are functions of a, r, and θ, respectively. We have used (5.153) as well as the results (5.126) and (5.127) obtained above for B'_0 and D'_0. If we take the limit of large a, then the expressions for B'_0 and D'_0 will be approximately similar to C'_0, being approximated by $e^{\pm\gamma a^2}$. Within this scenario, we also find that

$$\mathsf{A}_1' \sim \mathrm{e}^{-\gamma_1 a^2}\left[\Omega(r,\theta)\left\{\begin{array}{c}\mathcal{E}(a)\\ \mathcal{E}_{\mathrm{i}}(a)\end{array}\right\}+\beta\right], \tag{5.163}$$

$$\mathsf{E}_1' \sim \mathrm{e}^{\gamma_2 a^2}\left[\tilde{\Omega}(r,\theta)\left\{\begin{array}{c}\mathcal{E}(a)\\ \mathcal{E}_{\mathrm{i}}(a)\end{array}\right\}+\tilde{\beta}\right], \tag{5.164}$$

where $\mathcal{E}$ and $\mathcal{E}_{\mathrm{i}}$ are, respectively, the error function and error function with imaginary argument. These expressions still have to satisfy (5.154) and (5.156), where we can apply

$$\partial_\phi \sim \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial}{\partial r} - \mathrm{i}\frac{1}{r}\frac{\partial}{\partial\theta}\right).$$

This would lead to a system of partial differential equations in the r and θ variables that are first order and linear. Nevertheless, it is not obvious how to obtain exact or approximate solutions.

A power series solution (or a numerical approximation) is in principle feasible and should provide some information. The analysis of (5.158) and (5.159) can proceed along similar lines. However, given that they constitute (Wheeler–DeWitt-type) second-order partial differential equations, the situation is much more complicated.

Some satisfaction can be obtained, however, from the a-dependence of A_1' and E_1', as well as the form of C_1'. The ϕ and $\overline{\phi}$ dependence of A_1' and E_1' could in principle be found (even if approximated by leading terms in a series). Of course, by aiming at specific physical systems, e.g., wormhole [19] or no-boundary [18] configurations, we select boundary conditions or particular symmetry restrictions for the corresponding equations.

5.1.5 Vector Supermultiplets

Although also encouraging, bringing SQC closer to a complete and realistic scenario with the inclusion of a scalar supermultiplet, the previous sections also revealed complexities and difficulties which arise when one proceeds from a pure (vacuum) case to the scalar supermultiplet (see Sect. 3.3.1). Solutions were nevertheless found, and with properties associated with known bosonic states, such as the Hartle–Hawking (no-boundary) [18] or the Hawking–Page (wormhole) [19]. But to extract a full cosmological description, other ingredients remain to be added.

In this section, one such step is introduced. A vector supermultiplet, formed by gauge vector fields $v_\mu^{(a)} \equiv A_\mu^{(a)}$ and fermionic partners, is added [25–27, 42]. An FRW model solely in the presence of a vector supermultiplet will be discussed, with *non-trivial* solutions found in different fermionic sectors. Among these, we can identify (part of) the no-boundary solution [43] and another state which could to some extent be interpreted as a quantum wormhole state.

Ansatze for the Matter Variables

For simplicity, we take the gauge group $SU(2)$ as present in a $k = +1$ FRW model. An important consequence of not having scalar fields and their fermionic partners is that the Killing potentials $D^{(a)}$ (see Sect. 4.2.3) and related quantities are now absent. In fact, if we had complex scalar fields, a Kähler manifold could be considered, with metric $\mathsf{G}_{IJ^*} \equiv \mathsf{G}_{\mathsf{I}\bar{\mathsf{J}}}$ on the space of $(\phi^{\mathsf{I}}, \phi^{\mathsf{J}^*})$. For $SU(2)$ with $\phi = \overline{\phi} = 0$, this implies $D^{(1)} = D^{(2)} = 0$ and $D^{(3)} = -1/2$. However, since the $D^{(a)}$ are fixed up to constants which are now arbitrary, we can choose $D^{(3)} = 0$ consistently.

The simplest choice for the matter fields in an FRW geometry is as follows:

- In strictly bosonic quantum cosmology with gauge fields, it is not sufficient for $A_\mu^{(a)}$ to have simple homogeneous components. A special ansätze is required for $A_\mu^{(a)}$, depending on the gauge group considered, and this may then also affect the choices for ϕ. A suitable ansatz for $A_\mu^{(a)}$ requires it to be *invariant up to gauge transformations*. For our chosen gauge group $SO(3) \sim SU(2)$, the spin-1 field is taken to be [38, 43–54]

$$\mathsf{A}_\mu(t)\omega^\mu = \left[\frac{f(t)}{4}\varepsilon_{(a)i(b)}\mathcal{T}^{(a)(b)}\right]\omega^i \ , \tag{5.165}$$

  where $\{\omega^\mu\}$ represents the moving coframe $\{\omega^\mu\} = \{\mathrm{d}t, \omega^i\}$, $\omega^i = \hat{e}^i{}_{\hat{c}}\mathrm{d}x^{\hat{c}}$, $(i, \hat{c} = 1, 2, 3)$, of one-forms, invariant under the left action of $SU(2)$, and $\mathcal{T}_{(a)(b)}$ are the generators of the $SU(2)$ gauge group.
- The idea of this ansatz for a non-Abelian spin-1 field is to define a *homomorphism* of the isotropy group $SO(3)$ to the gauge group. This homomorphism defines the gauge transformation which *compensates* for the action of a given $SO(3)$ rotation of the local coordinate transformation group. Hence, the above form for the gauge field, where the A_0 component is taken to be identically zero. None of the gauge symmetries will survive: all the available gauge transformations are required to cancel out the action of a given $SO(3)$ rotation. Thus, we will *not*[6] have a gauge constraint $\mathcal{Q}^{(a)} = 0$.
- Fermions in simple minisuperspace models have also been considered in [55–62]. Some questions concerning the (in)consistency of these models were raised in [51], and an attempt to clarify them was made.[7] Similarly to the case where only scalar fields are present, for the vector supermultiplet we take a simple spin-1/2

[6] However, in the case of larger gauge group, some of the gauge symmetries will survive. These will give rise, in the one-dimensional model, to local internal symmetries with a reduced gauge group. Therefore, a gauge constraint can be expected to play an important role in that case. It would be particularly interesting to study such a model (see Chap. 8).

[7] The introduction of fermions in ordinary quantum cosmological models with gauge fields led to additional *non-trivial* ansätze for the fermionic fields [63]. These involve restrictions from group theory, rather than just imposing time dependence.

field (like χ_A) as fermionic partner for (5.165). The most general ansatz for the spin-1/2 fields is just to take

$$\lambda_A^{(a)} = \lambda_A^{(a)}(t) \;, \tag{5.166}$$

and correspondingly for its Hermitian conjugate with $(a) = 1, 2, 3$.

We will then use the ansätze (5.1), (5.2) together with the expression (5.165), taking $\lambda_A^{(a)} = \lambda_A^{(a)}(t)$. The action is reduced to one with a finite number of degrees of freedom. Starting from the action thereby obtained, we can retrieve the Hamiltonian formulation of this model.

Quantum Constraints

Let us obtain expressions for the quantum supersymmetry constraints. First we need to redefine the new fermionic fields, i.e., besides ψ_A, we also need to deal with $\lambda_A^{(a)}$ in order to simplify the Dirac brackets. Similarly to Sect. 5.1.1, for the $\lambda_A^{(a)}$ field we write

$$\hat{\lambda}^{(a)}{}_A \equiv \frac{\varsigma a^{3/2}}{2^{1/4}}\lambda^{(a)}{}_A \;, \qquad \hat{\bar{\lambda}}^{(a)}{}_{A'} \equiv \frac{\varsigma a^{3/2}}{2^{1/4}}\bar{\lambda}^{(a)}{}_{A'} \;, \tag{5.167}$$

giving

$$\pi_{\hat{\lambda}^{(a)}{}_A} = -\mathrm{i}n_{AA'}\hat{\bar{\lambda}}^{(a)A'} \;, \qquad \pi_{\hat{\bar{\lambda}}^{(a)}{}_{A'}} = -\mathrm{i}n_{AA'}\hat{\lambda}^{(a)A} \;, \tag{5.168}$$

with

$$\left[\hat{\lambda}^{(a)}{}_A, \hat{\bar{\lambda}}^{(b)}{}_{A'}\right]_{\mathrm{D}} = -\mathrm{i}\delta^{(a)(b)}n_{AA'} \;. \tag{5.169}$$

Furthermore,

$$\left[a, \pi_a\right]_{\mathrm{D}} = 1 \;, \qquad \left[f, \pi_f\right]_{\mathrm{D}} = 1 \;, \tag{5.170}$$

and the rest of the brackets are zero. In addition, dropping the hats, unprimed spinors are subsequently defined by

$$\bar{\lambda}^{(a)}{}_A \equiv 2n_A{}^{B'}\bar{\lambda}^{(a)}{}_{B'} \;, \tag{5.171}$$

with which the new Dirac brackets are

$$\left[\psi_A, \bar{\psi}_B\right]_{\mathrm{D}} = \mathrm{i}\varepsilon_{AB} \;, \qquad \left[\lambda^{(a)}{}_A, \bar{\lambda}^{(b)}{}_B\right]_{\mathrm{D}} = -\mathrm{i}\delta^{(a)(b)}\varepsilon_{AB} \;. \tag{5.172}$$

The rest of the brackets remain unchanged. Again, quantum mechanically, one replaces the Dirac brackets by anticommutators if both arguments are odd (O) or commutators if otherwise (E). The only non-zero (anti-)commutator relations are

$$\left\{\lambda^{(a)}{}_A, \overline{\lambda}^{(b)}{}_B\right\} = \delta^{(a)(b)}\varepsilon_{AB}\ , \qquad \left\{\psi_A, \overline{\psi}_B\right\} = -\varepsilon_{AB}\ , \tag{5.173}$$

$$\left[a, \pi_a\right] = \left[f, \pi_f\right] = \mathrm{i}\ . \tag{5.174}$$

Consequently, we choose $(\lambda_A^{(a)}, \psi_A, a, f)$ to be the coordinates of the configuration space, and $(\overline{\lambda}_{A'}^{(a)}, \overline{\psi}_{A'}, \pi_a, \pi_f)$ to be the momentum operators in this representation. Hence, we take

$$\overline{\lambda}^{(a)}{}_A \longrightarrow -\frac{\partial}{\partial\lambda^{A(a)}}\ , \qquad \overline{\psi}_A \longrightarrow \frac{\partial}{\partial\psi^A}\ ,$$
$$\pi_a \longrightarrow -\mathrm{i}\frac{\partial}{\partial a}\ , \qquad \pi_f \longrightarrow -\mathrm{i}\frac{\partial}{\partial f}\ . \tag{5.175}$$

When our spin 1/2 matter fields $\lambda_A^{(a)}$ are taken into account, the generalization of the J_{AB} constraint is

$$\mathcal{J}_{AB} = \psi_{(A}\overline{\psi}^{B'}n_{B)B'} - \lambda_{(A}^{(a)}\overline{\lambda}^{(a)B'}n_{B)B'} = 0\ . \tag{5.176}$$

One can justify it by observing either that it arises from the corresponding constraints of the full theory, or that its quantum version describes the invariance of the wave function under Lorentz transformations.

The contributions to the $\overline{\mathcal{S}}_{A'}$ constraint from the spin 1 field are

$$-\mathrm{i}\frac{\sqrt{2}}{3}\pi_f\sigma^a{}_{BA'}\lambda^{(a)B} + \frac{\varsigma^2a^3}{6}\sigma^a{}_{BA'}\lambda^{(a)B}n_{CB'}\left[\sigma^{bCC'}\overline{\psi}_{C'}\overline{\lambda}^{(b)B'} + \sigma^{bAB'}\psi_A\lambda^{(b)C}\right]$$
$$+\frac{1}{8\sqrt{2}}\varsigma^2a^4\sigma^{(a)C}{}_{A'}\left[1-(f-1)^2\right]\lambda^{(a)}{}_C$$
$$+\frac{1}{2}\varsigma^2a^3\lambda^{(a)A}\left[-n_{AB'}\overline{\psi}_{A'}\overline{\lambda}^{(a)B'} + n_{BA'}\psi_A\lambda^{(a)B}\frac{1}{2}n_{AA'}\psi_B\lambda^{(a)B}\right.$$
$$\left.+\tfrac{1}{2}n_{AA'}\overline{\psi}_{B'}\overline{\lambda}^{(a)B'}\right]\ . \tag{5.177}$$

The following terms may[8] also be present in the $\overline{\mathcal{S}}_{A'}$ supersymmetry constraint:

$$-\frac{1}{\sqrt{2}}\varsigma^2a^3gD^{(a)}n_{AA'}\lambda^{(a)A} - \frac{1}{4}\varsigma^2a^3\left[n_{AB'}\lambda^{(a)A}\overline{\lambda}^{(a)}{}_{A'}\overline{\psi}^{B'} + n_{AA'}\lambda^{(a)A}\overline{\lambda}^{(a)}{}_{B'}\overline{\psi}^{B'}\right]\ . \tag{5.178}$$

[8] In the present case, the first term is absent due to our choice of gauge group.

The supersymmetry constraint[9] $\overline{\mathcal{S}}_{A'}$ will include the sum of the above expressions, with the supersymmetry constraint $\mathcal{S}_A$ being just the complex conjugate of $\overline{\mathcal{S}}_{A'}$.

We put all the fermionic derivatives in $\mathcal{S}_A$ on the right, while all the fermonic derivatives $\overline{\mathcal{S}}_A$ are on the left. Implementing all these redefinitions, the supersymmetry constraints have the differential operator form

$$\begin{aligned}
\mathcal{S}_A = &-\frac{1}{2\sqrt{6}}a\psi_A\frac{\partial}{\partial a} - \sqrt{\frac{3}{2}}\varsigma^2 a^2\psi_A \\
&-\frac{1}{8\sqrt{6}}\psi_B\psi^B\frac{\partial}{\partial\psi^A} - \frac{1}{4\sqrt{6}}\psi^C\lambda^{(a)}_C\frac{\partial}{\partial\lambda^{(a)A}} \\
&+\frac{1}{3\sqrt{6}}\sigma^a{}_{AB'}\sigma^{bCC'}n_D{}^{B'}n^B{}_{C'}\lambda^{(a)D}\psi_C\frac{\partial}{\partial\lambda^{(b)B}} \\
&+\frac{1}{6\sqrt{6}}\sigma^a{}_{AB'}\sigma^{bBA'}n_D{}^{B'}n^E{}_{A'}\lambda^{(a)D}\lambda^{(b)}_B\frac{\partial}{\partial\psi^E} \\
&-\frac{1}{2\sqrt{6}}\psi_A\lambda^{(a)C}\frac{\partial}{\partial\lambda^{(a)C}} + \frac{3}{8\sqrt{6}}\lambda^a{}_A\lambda^{(a)C}\frac{\partial}{\partial\psi^C} \\
&+\sigma^a{}_{AA'}n^{BA'}\lambda^{(a)}_B\left\{-\frac{\sqrt{2}}{3}\frac{\partial}{\partial f} + \frac{1}{8\sqrt{2}}\left[1-(f-1)^2\right]\varsigma^2\right\}
\end{aligned} \tag{5.179}$$

and

$$\begin{aligned}
\overline{\mathcal{S}}_A = &\frac{1}{2\sqrt{6}}a\frac{\partial}{\partial a}\frac{\partial}{\partial\psi^A} - \sqrt{\frac{3}{2}}\varsigma^2 a^2\frac{\partial}{\partial\psi^A} \\
&-\frac{1}{8\sqrt{6}}\varepsilon^{BC}\frac{\partial}{\partial\psi^B}\frac{\partial}{\partial\psi^C}\psi_A + \frac{1}{4\sqrt{6}}\varepsilon^{BC}\frac{\partial}{\partial\psi^B}\frac{\partial}{\partial\lambda^{(a)C}}\lambda^{(a)}{}_A \\
&+\frac{1}{3\sqrt{6}}\sigma^{aB}{}_{A'}\sigma^{bCC'}n_A{}^{A'}n^D{}_{C'}\frac{\partial}{\partial\psi^D}\frac{\partial}{\partial\lambda^{(a)B}}\lambda^{(b)}{}_C \\
&+\frac{1}{6\sqrt{6}}\sigma^{aB}{}_{A'}\sigma_D{}^{bB'}n_A{}^{A'}n^C{}_{B'}\frac{\partial}{\partial\lambda^{(a)B}}\frac{\partial}{\partial\lambda^{(b)C}}\psi^D \\
&+\frac{1}{2\sqrt{6}}\frac{\partial}{\partial\psi^A}\frac{\partial}{\partial\lambda^{(a)B}}\lambda^{(a)B} + \frac{3}{8\sqrt{6}}\frac{\partial}{\partial\lambda^{(a)B}}\frac{\partial}{\partial\lambda^{(a)A}}\psi^B \\
&+n_A{}^{A'}\sigma^{aB}{}_{A'}\left\{\frac{2\sqrt{2}}{3}\frac{\partial}{\partial f} + \frac{1}{4\sqrt{2}}\left[1-(f-1)^2\right]\varsigma^2\right\}\frac{\partial}{\partial\lambda^{(a)B}}\ .
\end{aligned} \tag{5.180}$$

The Lorentz constraint $\mathcal{J}_{AB}$ implies that a physical wave function should be a Lorentz scalar. We can easily see that the most general form of the wave function is

[9] Notice that the above expressions correspond to a gauge group $SU(2)$ and hence a compact Kähler manifold, which implies that the analytical potential $\mathsf{P}(\phi^I)$ is zero.

$$\begin{aligned}\Psi = &\ \mathsf{A} + \mathsf{B}\psi^C\psi_C + \mathsf{d}_a\lambda^{(a)C}\psi_C + \mathsf{c}_{ab}\lambda^{(a)C}\lambda^{(b)}{}_C + \mathsf{e}_{ab}\lambda^{(a)C}\lambda^{(b)}{}_C\psi^D\psi_D \\ &+\mathsf{c}_{abc}\lambda^{(a)C}\lambda^{(b)}{}_C\lambda^{(c)D}\psi_D + \mathsf{c}_{abcd}\lambda^{(a)C}\lambda^{(b)}{}_C\lambda^{(c)D}\lambda^{(d)}{}_D \\ &+\mathsf{d}_{abcd}\lambda^{(a)C}\lambda^{(b)}{}_C\lambda^{(c)D}\lambda^{(d)}{}_D\psi^E\psi_E \\ &+\mathcal{D}_1\lambda^{(2)C}\lambda^{(2)}{}_C\lambda^{(3)D}\lambda^{(3)}_D\lambda^{(1)E}\psi_E \\ &+\mathcal{D}_2\lambda^{(1)C}\lambda^{(1)}{}_C\lambda^{(3)D}\lambda^{(3)}_D\lambda^{(2)E}\psi_E + \mathcal{D}_3\lambda^{(1)C}\lambda^{(1)}{}_C\lambda^{(2)D}\lambda^{(2)}{}_D\lambda^{(3)E}\psi_E \\ &+\mathsf{C}\lambda^{(1)C}\lambda^{(1)}{}_C\lambda^{(2)D}\lambda^{(2)}{}_D\lambda^{(3)E}\lambda^{(3)}{}_E \\ &+\mathsf{D}\lambda^{(1)C}\lambda^{(1)}{}_C\lambda^{(2)D}\lambda^{(2)}{}_D\lambda^{(3)E}\lambda^{(3)}{}_E\psi^F\psi_F\ ,\end{aligned} \tag{5.181}$$

where $\mathsf{A}, \mathsf{B}, \ldots, \mathsf{D}$ above are functions of a and f alone. It contains *all allowed combinations* of the fermionic fields and is the most general Lorentz invariant function we can write down.

Quantum States

The next step is to solve the supersymmetry constraints $\mathcal{S}_A\Psi = 0$ and $\overline{\mathcal{S}}_{A'}\Psi = 0$. Since the wave function (5.181) is of even order in fermionic variables and stops at order 8, the expressions $\mathcal{S}_A\Psi = 0$ and $\overline{\mathcal{S}}_A\Psi = 0$ will be of odd order in fermionic variables and stop at order 7. Moreover, as fermionic terms like ψ^A, χ^A, $\psi^A\chi^C\chi_C$, etc., are linearly independent Grassmannian quantities, the action of the quantum operators (5.179) and (5.180) on (5.181) will produce ten equations from $\mathcal{S}_A\Psi = 0$ and another ten equations from $\overline{\mathcal{S}}_A\Psi = 0$. These equations are simply the bosonic expressions associated with each of the fermionic terms ψ^A, χ^A, $\psi^A\chi^C\chi_C$, etc., and each bosonic expression is therefore equated to zero.

Among the equations derived from $\mathcal{S}_A\Psi = 0$, we obtain

$$-\frac{a}{2\sqrt{6}}\frac{\partial \mathsf{A}}{\partial a} - \sqrt{\frac{3}{2}}\varsigma^2 a^2\mathsf{A} = 0\ , \tag{5.182}$$

$$-\frac{\sqrt{2}}{3}\frac{\partial \mathsf{A}}{\partial f} + \frac{1}{8\sqrt{2}}\left[1-(f-1)^2\right]\varsigma^2\mathsf{A} = 0\ . \tag{5.183}$$

These equations correspond, respectively, to the terms linear in ψ_A and $\lambda^{(a)}_A$. Equations (5.182) and (5.183) give the dependence of A on a and f, respectively. Solving these equations leads to $\mathsf{A} = \hat{\mathsf{A}}(a)\tilde{\mathsf{A}}(f)$ with

$$\mathsf{A} = \tilde{\mathsf{A}}(f)\mathrm{e}^{-3\varsigma^2 a^2}\ , \tag{5.184}$$

$$\mathsf{A} = \hat{\mathsf{A}}(a)\exp\left[\frac{3}{16}\varsigma^2\left(-\frac{f^3}{3} + f^2\right)\right]\ . \tag{5.185}$$

A similar relation exists for the $\overline{\mathcal{S}}_A\Psi = 0$ equations. From the term

$$\psi^A\lambda^{(1)}_E\lambda^{(1)E}\lambda^{(2)}_E\lambda^{(2)E}\lambda^{(3)}_E\lambda^{(3)E}$$

in Ψ, these give for $\mathsf{D} = \hat{\mathsf{D}}(a)\tilde{\mathsf{D}}(f)$,

$$\mathsf{D} = \tilde{\mathsf{D}}(f)\mathrm{e}^{3\varsigma^2 a^2} , \tag{5.186}$$

$$\mathsf{G} = \hat{\mathsf{G}}(a) \exp\left[\frac{3}{16}\varsigma^2 \left(\frac{f^3}{3} - f^2\right)\right] . \tag{5.187}$$

We notice that in our case study, we are indeed allowed to completely determine the dependence of A and D with respect to the scale factor a and the *effective* conformal scalar field f.

The solution (5.186), (5.187) is included in the Hartle–Hawking [18] (no-boundary) solutions of [43], where a $k = 1$ FRW universe with Yang–Mills fields was employed in a *non*-supersymmetric quantum cosmological framework. In fact, we basically recover the solution (3.8a) in [43] if we replace $f \to f + 1$. As can be checked, this procedure constitutes the right choice according to the definitions employed there for $A_\mu^{(a)}$. The solution (5.186), (5.187) is also associated with an anti-self-dual solution of the Euclideanized equations of motion. However, it is important to emphasize that not all the solutions presented in [43] can be recovered here. In particular, the Gaussian wave function (5.186), (5.187), peaked around $f = 1$ (after implementing the above transformation), represents only one of the components of the wave function therein. The wave function is peaked around the two minima of the corresponding quartic potential. In our model, the potential terms correspond to a 'square root' of the potential present in [52].

The solution (5.184), (5.185) has the features of a (Hawking–Page) wormhole solution, but here extended for Yang–Mills fields, which nevertheless have *not yet been found* in ordinary quantum cosmology. However, in spite of (5.184), (5.185) being regular for $a \to 0$ and damped for $a \to \infty$, it may not be well behaved when $f \to -\infty$.

The equations obtained from the cubic and fifth-order fermionic terms in $\mathcal{S}_A\Psi = 0$ and $\overline{\mathcal{S}}_A\Psi = 0$ can be dealt with by multiplying them by $n_{EE'}$ and using the relation $n_{EE'}n^{EA'} = \varepsilon_{E'}{}^{A'}/2$. Notice that the $\boldsymbol{\sigma}_a$ matrices are linearly independent and orthogonal to the n matrix. We will see that such equations provide the a and f dependence of the remaining terms in Ψ. It is important to point out that the dependence of the coefficients in Ψ corresponding to cubic fermionic terms in a and $\phi, \overline{\phi}$ is *mixed* through several equations. However, in the present FRW minisuperspace with vector fields, the analogous dependence on a, f occurs in *separate* equations. The equations for cubic and fifth-order fermionic terms further imply that any possible solutions are neither the Hartle–Hawking nor a wormhole state. In fact, we would get $\mathsf{d}_{(a)} \sim a^5\hat{\mathsf{d}}_{(a)}(a)\tilde{\mathsf{d}}_{(a)}(f)$ and similar expressions for the other coefficients in Ψ, with a prefactor a^n, $n \neq 0$. Hence, from their a-dependence equations, these cannot be either a Hartle–Hawking or a wormhole state. They correspond to *another type* of solution, which could be obtained from the corresponding Wheeler–DeWitt equation, but with completely different boundary conditions.

Finally, it is worth noting that the Dirac bracket of the supersymmetry constraints (5.179), (5.180) induces an expression whose bosonic sector corresponds to the gravitational and vector field components of the Hamiltonian constraint present in

[43]. Hence, as expected, the results (5.186) and (5.187) are consistent within the context of $N = 1$ SUGRA as a square-root of gravity.

5.2 Bianchi (Class A) Models

This section will be of particular importance, although the reader may be wondering why, since it looks at first glance just like the obvious extension proceeding from spatial homogeneity and isotropy towards a spatially anisotropic background. However, significant features will emerge here. Let us be more precise.

The FRW models provided adequate means for introducing and illustrating the methodology of SQC, for didactic purposes. However, it is a simplified framework. Additional degrees of freedom from spatial anisotropy (even if some dynamical process such as inflationary expansion would dilute their relevance) may do more than just increase the count of physical variables. Could they play a distinctive role in a SUSY very early universe, perhaps narrowing the gap between SQC and a full canonical quantum $N = 1$ SUGRA (and superstrings)?

In fact, it could be from the presence of these very degrees of freedom that SQC would attain a realistic validation in view of the fundamental results in [12]. We will recover an attractive connection between the contents of Sect. 4.1 of Vol. II and the spectrum of states of Bianchi-type SQC.

The more realistic context of Bianchi-type SQC does not come without a price, as the constraint equations become more complicated to deal with. Specific techniques are required, and it is important to learn about them since *open problems* within Bianchi SQC do indeed remain.

This is why we include a certain level of detail in this section, for the case of spatially anisotropic models in SQC, within a fermionic differential operator representation,[10] together with $N = 4$ SUSY inherited from $N = 1$ SUGRA. Hopefully, the reader will feel rather contented by the end. His or her perspective on the very early universe, both quantum mechanical and supersymmetric, will be significantly enlarged, and a connection with the essential building blocks of canonical (quantum) $N = 1$ SUGRA will be established, with plenty of other frontiers to push back and directions to venture in.

5.2.1 Minisuperspace Reduction Method

Two routes can be followed to construct supersymmetric Bianchi quantum cosmological models:

- The apparently more direct line is to follow the approach of Sect. 5.1.1 and substitute a specific Bianchi ansatz (consistent with the symmetries of the Bianchi model) directly into the classical action, thereby obtaining a reduced minisuperspace model and then quantizing it:

[10] Other approaches and frameworks will be described in the next chapter and in Vol. II.

  - However, in contrast with the simplicity obtained with FRW cosmologies, we would soon encounter a rather complicated framework.
  - The Bianchi models have more spatial anisotropy degrees of freedom and therefore require *more* fermionic modes. It turns out that simply imposing spatial homogeneity would *not* immediately lead to an ansatz invariant under homogeneous supersymmetry transformations.
  - This invariance is *only* achieved with Bianchi *non*-diagonal models, and by *combining* supersymmetry, coordinate, and Lorentz transformations.[11] It is only in this combined setting that we can retrieve a reduced Bianchi supersymmetric action and then a minisuperspace, inheriting invariance under local time translations, supersymmetry, and Lorentz transformations.

- A less obvious but computationally more efficient approach is to employ the quantum constraints *directly* from the general theory, i.e., use (4.93), (4.94), (4.97), and (4.98) for $\mathcal{J}_{AB}\Psi = 0$, $\overline{\mathcal{J}}_{A'B'}\Psi = 0$, $\mathcal{S}_A\Psi = 0$, $\overline{\mathcal{S}}_{A'}\Psi = 0$, and evaluate them *subject to* a simple Bianchi ansatz. This has often been applied in the framework of *diagonal* Bianchi models [1, 64], retrieving the constraints subject to the corresponding spatially homogeneous ansatz on the supergravity physical variables [65, 66] (see, e.g., [33, 67–69] and in particular [70], since it is quite enlightening on this matter):
  - In this way, the action of the supersymmetry constraints (4.97) and (4.98) on a wave function $\Psi(e^{AA'}{}_i, \psi^A{}_i)$ or equivalently, a wave function $\overline{\Psi}(e^{AA'}{}_i, \overline{\psi}^{A'}{}_i)$, is gauge invariant. These gauge invariant supersymmetry constraints annihilate a gauge invariant wave function.
  - Equations can then be obtained from the condition of a diagonal Bianchi IX metric, for example, in these equations.
  - In this case, it is important to note[12] that there is *no* loss of physical information. There are *no* off-diagonal components in the metric, and subsequently no off-diagonal terms are present either (the reader should consult [70] for more details).

5.2.2 Applying the Lorentz Constraint

Concerning the Lorentz constraint annihilation condition on the quantum states, some discussion is required (see Notes 5.3, 5.5, and 5.7). In most of the published work (see [70, 33] and references therein), the annihilation of the Lorentz constraint may not have been tackled thoroughly enough to satisfy some researchers (see Chap. 7 of Vol. II), although the models are described in the Hamiltonian formalism and the gauges are fixed in different, consistent ways. The widely

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[11] See Sect. 4.1.3 concerning the Teitelboim procedure.

[12] Moreover, it should be noticed that ${}^{(3s)}D_j$, the spatial covariant derivative on the gravitino, is explicitly present in the supersymmetry constraints (see Appendix A).

used and simplified proposal (within the differential operator representation for fermionic momenta) is to write the quantum state in the form of a power series expansion in the bosonic and fermionic variables, constructing all possible *invariants* from even combinations of them. Lorentz invariance would be automatically satisfied as far as *all* indices are contracted, thus retrieving a *scalar* wave function in effective terms. An important feature in such a quantum mechanical description is that the fermion number for some minisuperspaces can be conserved, since each fermionic sector may be treated separately thanks to their linear independence.

However, some researchers (see Chap. 7 of Vol. II) have pointed out an apparent weakness: what we have is a set of Lorentz constraints resulting from the canonical formulation of the theory, represented as differential operators (or in a matrix form), each of them annihilating the wave function. Their claim is that the method in this section does *not* use the Lorentz constraint unambiguously. A suggestion was made (a) to avoid discussing what the wave function should look like, how it should be expanded in Lorentz invariant pieces, and how many such pieces there are, and (b) in the context of the matrix representation (see Chap. 7 of Vol. II), to avoid solving *all* the Lorentz constraints for a diagonal Bianchi IX model. Hence, more care is needed in discussing how the wave function could and should be expanded in Lorentz invariant pieces, involving the spatial components of the tetrad $e_i^{AA'}$ and of the gravitino ψ_i^A, contracting all spatial and spinorial indices.

Note 5.9 Regarding the FRW case, there is an important distinctive feature, as we have been suggesting. In fact, we retrieved an $N = 4$ supersymmetric FRW minisuperspace from a dimensional reduction applied to the action of 4D $N = 1$ supergravity theory. Let us emphasize that it uses an integration over the spatial hypersurfaces, *eliminating* the presence of any spatial indices. In fact, in the action of the one-dimensional reduced mechanical FRW model, *only* spinorial indices are present. This rather specific property of supersymmetric minisuperspaces is a consequence of using a spatially homogeneous and isotropic FRW ansatz, where the physical bosonic and fermionic degrees of freedom are *severely restricted*. So with fewer (and less complicated) constraints, it is possible to find solutions for an FRW scenario, whereas in the full theory the arguments and results of [12, 13] in Vol. II would apply.
It should also be stressed that, in this FRW procedure, we did *not* use the constraints present in the full theory of quantum supergravity. The subsequent algebra and corresponding symmetries can be identified ab initio from the reduced minisuperspace model, and not from any expression imported from the full theory.

When the Lorentz annihilation in the wave function has been performed, solving the corresponding equations, we get a set of expressions that must still satisfy the supersymmetry constraints. This is usually achieved by solving a subsequent set of (simple) first-order differential equations (equivalent to a Dirac-like square root of the second order Wheeler–DeWitt equation in purely bosonic quantum cosmology).

5.2.3 Supersymmetric Vacuum Case

The development of Bianchi SQC is rather interesting, because quite apart from the reasons indicated in Sects. 5.2.1 and 5.2.2, and the implications and results we will explain and retrieve below, it is a story of endeavor and persistence. Let us explain briefly.

Using a triad ADM canonical formulation, Bianchi class A models obtained from pure $N = 1$ SUGRA have been studied in [33, 64, 71, 72, 66]. Quantum states were described by a wave function of the form $\Psi[e_{AA'i}, \psi_{Ai}]$ where $e_{AA'i}$ and ψ_{Ai} denote the two-component spinor form of the tetrad and the spin-3/2 gravitino field, respectively. The wave function was expanded in even powers of ψ_{Ai} up to sixth order, because of the anticommutation relations of the six spatial components of the gravitino fields (see Sect. 5.2.2), in a Lorentz invariant manner, symbolically represented[13] by ψ^0, ψ^2, ψ^4, up to ψ^6. However, solutions were *only* present in the empty ψ^0 (bosonic) and fermionic filled ψ^6 sectors! Furthermore, we could *not* identify both a wormhole (Hawking–Page) [19] and a no-boundary (Hartle–Hawking) [18] state in the same spectrum of solutions. Finding one *or* the other depended on homogeneity conditions imposed on the gravitino [66].

In the following, we explain in more detail the process briefly described above.

Quantum States: A Partial Spectrum

Let us consider for illustrative purposes a diagonal Bianchi type IX model (see, e.g., [2]) whose 4-metric is given by $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where η_{ab} is the Minkowski metric and the non-zero components of the tetrad $e^a{}_\mu$ are given by

$$\begin{aligned} &e^0{}_0 = \mathcal{N}\ , \quad e^1{}_0 = a_1 \mathcal{N}^i e^1{}_i\ , \qquad e^2{}_0 = a_2 \mathcal{N}^i e^2{}_i\ , \qquad e^3{}_0 = a_3 \mathcal{N}^i e^3{}_i\ , \\ &e^1{}_i = a_1 e^1{}_i\ , \qquad e^2{}_i = a_2 e^2{}_i\ , \qquad e^3{}_i = a_3 e^3{}_i\ , \end{aligned} \tag{5.188}$$

with $e^1{}_i, e^2{}_i, e^3{}_3$ $(i = 1, 2, 3)$ a spatial triad basis of left-invariant one-forms on the unit three-sphere and $\mathcal{N}, \mathcal{N}^i, a_1, a_2$, and a_3 spatially constant. We thus write

$$h_{ij} = a_1^2 e^1{}_i e^1{}_j + a_2^2 e^2{}_i e^2{}_j + a_3^2 e^3{}_i e^3{}_j\ . \tag{5.189}$$

For the torsion-free connections (see Appendix A), we write

[13] In the representation $\overline{\Psi}(e^{AA'}{}_i, \overline{\psi}^{A'}{}_i)$, these have of the form $\overline{\psi}^6, \overline{\psi}^4, \overline{\psi}^2$, and $\overline{\psi}^0$, the superscript denoting the fermionic order.

$$\begin{aligned}\omega_{ABi}n^A{}_{B'}e^{BB'j} &= \frac{\mathrm{i}}{4}\left(\frac{a_3}{a_1a_2}+\frac{a_2}{a_3a_1}-\frac{a_1}{a_2a_3}\right)e^1{}_ie^{1j}\\ &+\frac{\mathrm{i}}{4}\left(\frac{a_1}{a_2a_3}+\frac{a_3}{a_2a_1}-\frac{a_2}{a_3a_1}\right)e^2{}_ie^{2j}\\ &+\frac{\mathrm{i}}{4}\left(\frac{a_2}{a_3a_1}+\frac{a_1}{a_2a_3}-\frac{a_3}{a_1a_2}\right)e^3{}_ie^{3j}\ . \end{aligned} \tag{5.190}$$

The differential equations obeyed by the wave function are found by employing the quantum constraints of the full theory of supergravity and (4.93), (4.94), (4.97), and (4.98) (see Chap. 4).

The supersymmetry and Lorentz transformation properties are all that is required of a physical state. Notice that $\mathcal{H}_{AA'}$ follows from the anticommutator $\{\mathcal{S}_A, \overline{\mathcal{S}}_{A'}\} \sim \mathcal{H}_{AA'}$, which differs classically from this one by terms linear in $\mathcal{J}^{AB}$ and $\overline{\mathcal{J}}^{A'B'}$ (see Sect. 4.2.2).

Henceforth, the framework of Sect. 4.2.2 is employed subject to a *simple* Bianchi ansatz[14] on the spatial variables $e^{AA'}{}_i$, $\psi^A{}_i$, and $\overline{\psi}^{A'}{}_i$. We further require $\psi^A{}_0$, $\overline{\psi}^{A'}{}_0$ to be functions of time alone and $\psi^A{}_i$ and $\overline{\psi}^{A'}{}_i$ to be spatially homogeneous in the basis $e^a{}_i$. Equivalently, $\psi^A_i e^{BB'i}$ and $\overline{\psi}^{A'}{}_i e^{BB'i}$ are functions of time alone. The purpose is to obtain a wave function $\Psi(e^{AA'}{}_i, \psi^A{}_i)$, through an expansion in even powers of $\psi^A{}_i$ (i.e., as indicated above, with ψ^0, ψ^2, ψ^4, and ψ^6). This can be achieved by means of a series of ingredients, namely the decomposition $\psi_A{}^i e_{BB'i} = \psi_{ABB'}$, with

$$\psi_{ABB'} = -2n^C{}_{B'}\gamma_{ABC} + \frac{2}{3}(\beta_A n_{BB'} + \beta_B n_{AB'}) - 2\varepsilon_{AB}n^C{}_{B'}\beta_C\ , \tag{5.191}$$

where $\gamma_{ABC} = \gamma_{(ABC)}$ is totally symmetric. The β^A and γ^{BCD} spinors constitute the spin 1/2 and 3/2 modes of the gravitino fields, when these are split into irreducible representations of the Lorentz group (see Sect. A.3). As the Lorentz constraints imply that Ψ is invariant under Lorentz transformations, a *possible* ansatz for Ψ satisfying $\mathsf{J}_{AB}\Psi = 0$, would be

$$\begin{aligned}\Psi &= \mathsf{A} + \mathsf{B}\beta_A\beta^A + \mathsf{C}\gamma^{BCD}\gamma_{BCD} + \mathsf{D}\beta_A\beta^A\gamma^{BCD}\gamma_{BCD}\\ &+\mathsf{E}\left(\gamma^{BCD}\gamma_{BCD}\right)^2 + \mathsf{F}\beta_A\beta^A\left(\gamma^{BCD}\gamma_{BCD}\right)^2\ ,\end{aligned} \tag{5.192}$$

where A to F are functions of a_1, a_2, a_3. The first term in (5.192) corresponds to the bosonic ψ^0 part, while the second and third terms in Ψ represent the quadratic

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[14] Meaning that only time dependence is required for the effective degrees of freedom (see the discussion in Sect. 5.1.5). It should be noted that this simple ansatz is *not* invariant under homogeneous supersymmetry transformations. To obtain an ansatz that is invariant under supersymmetry, one must use a *non*-diagonal triad $e^a{}_i = b^a{}_b E^b{}_i$, where b_{ab} is symmetric ($a, b = 1, 2, 3$ here), *combined with* supersymmetry, homogeneous spatial coordinate, and local Lorentz transformations.

sectors. Similarly, the fourth and fifth correspond to the quartic sector, and the last term in (5.192) is just the fermionic filled sector.[15]

In a summarized manner, the required steps are:

- Consider first the (bosonic) ψ^0 part $\mathsf{A}(a_1, a_2, a_3)$ of the wave function. It automatically obeys the constraint $\mathcal{S}_A \Psi = 0$, since this involves differentiation with respect to $\psi^A{}_i$. The only remaining constraint is $\overline{\mathcal{S}}_{A'} \Psi = 0$. For a *diagonal* Bianchi IX geometry, this constraint reads

$$\varepsilon^{ijk} e_{AA'i} {}^{(3s)}\omega^A{}_{Bj} \psi^B{}_k \mathsf{A} - \frac{1}{2} \mathsf{k}^2 \psi^A{}_i \frac{\delta \mathsf{A}}{\delta e^{AA'}{}_i} = 0 \,. \tag{5.193}$$

- For the torsion-free connections, use

$$ {}^{(3s)}\omega^{00}{}_i = X_i + \mathrm{i} Y_i \,, \qquad {}^{(3s)}\omega^{11}{}_i = X_i - \mathrm{i} Y_i \,, \tag{5.194}$$

$$ {}^{(3s)}\omega^{01}{}_i = {}^{(3s)}\omega^{01}{}_i = \frac{1}{4} \mathrm{i} \left(-\frac{a_1}{a_2} - \frac{a_2}{a_1} + \frac{a_3^2}{a_1 a_2} \right) e^3{}_i \,, \tag{5.195}$$

where

$$X_i = \frac{1}{4} \left(\frac{a_3}{a_1} + \frac{a_1}{a_3} - \frac{a_2^2}{a_1 a_3} \right) e^2{}_i \,, \qquad Y_i = \frac{1}{4} \left(\frac{a_3}{a_2} + \frac{a_2}{a_3} - \frac{a_1^2}{a_2 a_3} \right) e^1{}_i \,. \tag{5.196}$$

- Since the homogeneous fields $\psi^A{}_i$ are otherwise arbitrary, they may be cancelled in (5.193). Using (A.122) and (A.123), then (since $\mathsf{k}^2 = 8\pi G$)

$$\varepsilon^{ijk} e^{(3s)}_{AA'i} \omega^A{}_{Bj} \psi^B{}_k \mathsf{A} + (8\pi G) e_{BA'j} \frac{\delta \mathsf{A}}{\delta h_{jk}} = 0 \,. \tag{5.197}$$

- One can then contract $e^{BA'm}$ in (5.193) to obtain the equation

$$\varepsilon^{ijk} e_{AA'i} e^{BA'm\,(3s)} \omega^A{}_{Bj} \psi^B{}_k \mathsf{A}[h_{jk}] - \frac{1}{2} \mathsf{k}^2 \psi^A{}_i \frac{\delta \mathsf{A}}{\delta h_{in}} = 0 \,, \tag{5.198}$$

  and use (A.50) and (A.51).
- Contracting with allowed variations δh_{km} of diagonal Bianchi IX 3-metrics,

$$\delta h_{ij} = \frac{\partial h_{ij}}{\partial a_1} = 2 a_1 e_i^1 e_j^1 \,, \tag{5.199}$$

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[15] A term $(\beta^A \gamma_{ABC})^2 = \beta^A \gamma_{ABC} \beta^D \gamma_D{}^{BC}$ can be rewritten, using the anticommutation of the β and γ terms, as $\beta^E \beta_E \varepsilon^{AD} \gamma_{ABC} \gamma_D{}^{BC} \sim (\beta_E \beta^E)(\gamma_{ABC} \gamma^{ABC})$. Similarly, any quartic in γ_{ABC} can be rewritten as a multiple of $(\gamma_{ABC} \gamma^{ABC})^2$. Since there are only four independent components of $\gamma_{ABC} = \gamma_{(ABC)}$, only one independent quartic can be made from γ_{ABC}, and it is sufficient to check that $(\gamma_{ABC} \gamma^{ABC})^2$ is non-zero. Now $\gamma_{ABC} \gamma^{ABC} = 2\gamma_{000}\gamma_{111} - 6\gamma_{100}\gamma_{011}$. Hence $(\gamma_{ABC} \gamma^{ABC})^2$ includes a non-zero quartic term $\gamma_{000}\gamma_{100}\gamma_{110}\gamma_{111}$ [2].

and integrating over three-geometries (to get $\partial A/\partial a_1$), this gives

$$\frac{\partial(\ln \mathsf{A})}{\partial a_1} = \int \mathrm{d}^3x \frac{\delta(\ln \mathsf{A})}{\delta h_{km}(x)} \frac{\partial h_{km}(x)}{\partial a_1} = -\mathsf{k}^{-2} a_1 \int \mathrm{d}^3 x h^{1/2} \ , \tag{5.200}$$

where $\int \mathrm{d}^3 x h^{1/2} = 16\pi^2$ is here the volume of the compact 3-space with $a_1 = a_2 = a_3 = 1$. Similar expressions for $\partial(\ln \mathsf{A})/\partial a_2$ and $\partial(\ln \mathsf{A})/\partial a_3$ lead to the wave function $\mathsf{A}(a_1, a_2, a_3)$:

$$\mathsf{A} \sim \exp(-I) \ , \qquad I = \pi \left(a_1^2 + a_2^2 + a_3^2\right) \ . \tag{5.201}$$

- Using equation $\mathcal{S}_A \overline{\Psi} = 0$, one gets $\mathsf{E} \sim \exp(I)$. Hence the bosonic and fermionic filled states of the theory both have very simple *semi-classical* forms.
- With respect to the quadratic and quartic fermionic components of Ψ, with the ansatz (5.192), a similar computational procedure is employed. However, an unimaginable (at this point!) consequence of using all the equations implied by the supersymmetry constraints was that *no* states are possible in the intermediate sectors of ψ^2 and ψ^4 order (see Exercise 5.6).

It should be stressed that the above conclusions are easily extended to Bianchi class A models [73].[16] More precisely, the physical states are, respectively, given by

$$\psi^0 \longrightarrow h^{s/2} \exp\left(\frac{1}{2}\zeta m^{pq} h_{pq}\right) \ , \quad \psi^6 \longrightarrow h^{-s/2} \exp\left(-\frac{1}{2}\zeta m^{pq} h_{pq}\right) \Pi_i (\psi_i^A)^2 \ . \tag{5.202}$$

Here h is $\det(h_{pq})$ and m^{pq} is defined from the relation

$$\mathrm{d}\omega^p = \frac{1}{2} m^{pq} h^{-\frac{1}{2}} \varepsilon_{qrs} \omega^r \otimes \omega^s \ ,$$

where ω^r are a basis of left-invariant 1-forms on the spacelike hypersurface of homogeneity. The constant symmetric matrix m^{pq} is fixed by the chosen Bianchi type [74, 75]. In addition, the parameter s specifies the *general* ambiguity of the operator ordering arising from the non-commutativity of ψ_i^A, $\overline{\psi}_j^{A'}$, and $p^{AA'}{}_k$.

The simple semi-classical form $\mathsf{A} \sim e^{-I}$ represents a (Hawking–Page) wormhole quantum state [19]. It is certainly regular at small 3-geometries, and decays away rapidly at large 3-geometries. Moreover, I is the Euclidean action of an asymptotically Euclidean 4D classical solution, outside a 3-geometry with metric (5.189), as confirmed by studying the Hamilton–Jacobi equations [64, 72]. These give the classical flow corresponding to the action I. However, $E \sim \mathrm{e}^I$ is *not* the no-boundary (Hartle–Hawking) state [18]. This conclusion can be reached by checking that $-I$ is the action of a regular Riemannian solution of the classical field equations, with metric (5.189) prescribed on the *outer* boundary. It is quite

---

[16] Only the class A models allow the spatial sections to be compactified by factoring if necessary by a discrete subgroup of isometries [74, 76].

satisfactory that the solution $\exp(-I)$ gives a wormhole state for Bianchi IX, but it seems very strange that the Hartle–Hawking state is *not* allowed by the quantum constraints above. (But see Exercise 5.5. It seems that including *another* definition of spatial homogeneity brings up this state [66].)

The curious and intriguing findings above were joined by yet another disturbing result. When a cosmological constant ($\Lambda < 0$) was added, it led to the undesirable situation that there were *no* physical states except the trivial one $\Psi = 0$ [33] (see Exercise 5.6).[17] It seemed that the gravitational and gravitino modes that were allowed to be excited contributed in such a way as to give only very simple states or even forbid any physical solutions. Furthermore, these solutions were shown [12] to have no counterpart in the full theory: states with a *finite* number of fermions are *impossible* there. These results seemed then to suggest that minisuperspaces might turn out to be useless as models for full supergravity.

Towards the Complete Spectrum

The results discussed in the last section were quite disconcerting. In fact, the subject of SQC was put to a harsh test and its future seemed in doubt. However, subsequent research, which can be found in [68, 69, 77] and also [70, 2], provided the required breakthrough.

In fact, the reason for the problem was the expansion of the Lorentz-invariant wave function $\Psi[e_{AA'i}(x), \psi^i_A(x)]$ with (5.191). Lorentz invariants were constructed *only from* the irreducible spin-1/2 and spin-3/2 components contained in the Rarita–Schwinger field $\psi_i{}^A$. There are only two such invariants that are bilinear and two more that are quadri-linear in $\psi_i{}^A$. But there are indeed further invariants,[18] involving irreducible spin-2 components contained in the gravitational degrees of freedom. In fact, in Bianchi models where $m^{ij} \neq 0$, such components can be used to generate up to $\binom{6}{2} = 15$ Lorentz invariants in the 2-fermion sector, e.g., $m^{ij}\psi_i{}^A\psi_{jA}$ or $m^{pr}h_{rs}m^{sq}\psi_p{}^A\psi_{qA}$, bearing in mind that we are taking $i, j, \ldots, m, \ldots, q = 1, 2, 3$ for the spatial reduction of world indices. Hence, the (Lorentz-invariant) component ansatz for solutions in the 2- and 4-fermion sector in (5.191), which allowed for two Lorentz invariant amplitudes only, rather than the 15 permitted ones, was quite restrictive.

Nevertheless, the approach provided by (5.191), although incomplete, has its merits. In particular, it gave an insight into the computational peculiarities where spatial anisotropy degrees of freedom are present, which are then extended into the 15 Lorentz invariants. Moreover, although the method of [68, 69] approached the more realistic and broader context of Sect. 4.1, Vol. II, bringing *new* solutions, the

---

[17] Regarding the $k = +1$ FRW model, a bosonic state was found, namely the Hartle–Hawking solution for an anti-de Sitter case (see Sect. 5.1.2).

[18] For the case of an FRW model without supermatter, and due to the restriction of the gravitino field to its spin-1/2 mode component, the *former* ansatz for the wave function remains valid.

structure was subsequently much improved in [70], creating a clearer framework and even providing a way of probing situations where chirality is broken.

We start by presenting the framework introduced by [68, 69], which has the advantage of being formulated in a generic Bianchi class A setting from the start. However, it has its weaknesses. Each of the two functionals f and g will stand in for the 15 possible allowed amplitudes, and it is *not* suitable to deal with situations where chirality is broken (meaning that there is a mixing of wave function components through the coupling recovered by using the SUSY constraints). This is overcome in [70, 2].

Note 5.10 The commutators $[\overline{\mathcal{S}}_{B'}, \mathcal{H}_{AA'}]$ and $[\mathcal{S}_B, \mathcal{H}_{AA'}]$, which are proportional to Lorentz generators, constitute one of the essential new ingredients, on which all of the following is based.
The *other* ingredient is the *modified momentum* for the tetrad [68], inspired by the formulation in [78–81], where the use of the time gauge $e_i^0 = 0$ and insertion of additional boundary terms in the action of $N = 1$ SUGRA made it invariant under a left invariant subalgebra, with the benefit of *simpler* Dirac brackets for both the physical variables and then the theory constraints.

From the algebra of constraints, it is sufficient to demand that physical states are Lorentz scalars and annihilated by $\mathcal{S}_A$ and $\overline{\mathcal{S}}_{A'}$. Their annihilation by $\mathcal{H}_{AA'}$ is then automatically guaranteed. Equally important, the form of the constraint operators guarantees that physical states have a fixed fermion number $\mathcal{F} \equiv \psi_i{}^A \partial/\partial\psi_i^A$, given by the number of factors of $\psi_i{}^A$ in the ψ-representation: $\mathcal{F}$ must be an even number in Lorentz invariant states, and ranges from 0 to 6 in the present models. The physical states in the sectors $\mathcal{F} = 0$ and $\mathcal{F} = 6$ are easily obtained, and are given respectively by (reinstating $\hbar$)

$$\begin{aligned} \mathsf{A} &\sim \exp\left(\frac{V}{2\hbar} m^{pq} h_{pq}\right) \\ \mathsf{F} &= h \exp\left(-\frac{V}{2\hbar} m^{pq} h_{pq}\right) \prod_r (\psi_r)^2 \ , \end{aligned} \tag{5.203}$$

reproducing what is above. In addition, and with some detail to guide the reader:

- In the 2-fermion sector, let us consider the ansatz

$$\Psi_2 \mapsto \overline{\mathcal{S}}_{A'} \overline{\mathcal{S}}^{A'} f(h_{pq}) \ , \tag{5.204}$$

  where we require, of course, that $\overline{\mathcal{S}}_{A'} \overline{\mathcal{S}}^{A'} f \neq 0$. This new ansatz for the quadratic fermionic sector (see below for ψ^4) brings *new* Lorentz invariants to Ψ, e.g., $m^{pq} \psi_p{}^A \psi_{qA}$. Here f is a function of the h_{pq} alone, and hence, like $\overline{\mathcal{S}}_{A'} \overline{\mathcal{S}}^{A'}$, a Lorentz scalar. Therefore (5.204) *automatically* satisfies the Lorentz constraints and the $\overline{\mathcal{S}}$ constraints. The only remaining condition is $\mathcal{S}_A = 0$ applied to (5.204), which reduces to

$$\left[\mathcal{H}_{AA'}\overline{\mathcal{S}}^A\right]f + 2\overline{\mathcal{S}}^A\mathcal{H}_{AA'}f = 0\ . \tag{5.205}$$

The first term is proportional to $\mathcal{J}_{AB}$, $\overline{\mathcal{J}}_{A'B'}$, and $\overline{\mathcal{S}}_{C'}$, whence only $\overline{\mathcal{S}}_{C'}$ contributes, due to f being Lorentz invariant by construction. Factorizing $\overline{\mathcal{S}}_{C'}$ to the left, (5.205) leads to the Wheeler–DeWitt equation

$$\left[\mathcal{H}_{AA'}{}^{(0)} - \frac{\hbar^2}{16\pi^2 h^{1/2}}n^{AA'}\right]f(h_{pq}) = 0\ , \tag{5.206}$$

where $\mathcal{H}_{AA'}{}^{(0)}$ consists of bosonic terms of $\mathcal{H}_{AA'}$ which remain if the terms in $p_i^{AA'}$ are brought to the *right* and then set to zero.

- With respect to the 4-fermion sector, the following has been proposed:

$$\Psi_4 \longmapsto \mathcal{S}^A\mathcal{S}_A g(h_{pq})\prod_{r=1}^{3}(\psi_r)^2\ . \tag{5.207}$$

The wave function $\mathcal{S}^A\mathcal{S}_A g(h_{pq})$ automatically satisfies the Lorentz constraints and the $\mathcal{S}$ constraint. It remains to satisfy the $\overline{\mathcal{S}}_{A'}$ constraint, which reduces to

$$\mathcal{H}_{AA'}{}^{(1)}g(h_{pq}) = 0\ , \tag{5.208}$$

where $\mathcal{H}_{AA'}{}^{(1)}$ consists of the terms of $\mathcal{H}_{AA'}$ with the momentum brought to the left.

- Any solution of these Wheeler–DeWitt equations, which may be specified further by imposing, e.g., no-boundary, tunneling, or wormhole boundary conditions, generates a solution in the 2- and 4-fermion sectors.

Note 5.11 However, in spite of allowing a wider set of solutions, we may ask whether it is in fact fully general:

- *Only* two functionals f and g are used, whereas the full range gives the possibility of 15 associated with the 2-fermion sector plus 15 more corresponding to the 4-fermion sector. In other words, all the sets of 15 corresponding amplitudes of the, e.g., two-fermion sector are *assembled* within a single amplitude which must satisfy a Wheeler–DeWitt equation. This corresponds to a choice of the initial state of the Rarita–Schwinger field, and, in addition, just as much freedom in the choice of initial conditions as the Wheeler–DeWitt equation of the corresponding Bianchi models in pure gravity.

- As a consequence, nontrivial solutions depending on the factor ordering, i.e., the (Hawking–Page) wormhole solution and the (Hartle–Hawking) no-boundary state were then found for *all* the fermionic sectors, of which infinitely many have fermion number $\mathcal{F} = 2$ and $\mathcal{F} = 4$. Furthermore, the new physical states we find in the minisuperspace models are direct analogues of physical states in *full* supergravity. (While the states in the empty and filled sectors would span at most a 2D Hilbert space, these physical states identified in the middle fermionic sectors would span an infinite-dimensional Hilbert space, just as in the Bianchi models of pure gravity.) Finally, supersymmetric minisuperspace models recover their significance as models of full supergravity.
- A generalization of these solutions for the case of full supergravity was provided in [82] and is briefly described in Chap. 4 of Vol. II. The algebra of the constraints has a similar form in full supergravity. While the states in the empty and filled sectors would span at most a 2D Hilbert space, these physical states identified in the middle fermionic sectors would span an infinite-dimensional Hilbert space, just as in the Bianchi models of pure gravity.
- In fact, if the algebra of the local generators of the constraints still has the form it has for the homogeneous case, physical states like Ψ_2 and Ψ_4 with products of $(\overline{\mathcal{S}})^2$ or $(\mathcal{S})^2$ would exist over all points of the spacelike 3-surface, thus leading to states with infinite fermion number. How those Hilbert spaces ought to be constructed is an *open issue*.

Let us now describe a more thorough investigation of these middle states in the fullest possible detail and overcome the weakness in the methodology of the above framework. We recall that this involves reducing and solving a Wheeler–DeWitt equation for two functionals, together with the ansatz $\overline{\mathcal{S}}_{A'}\overline{\mathcal{S}}^{A'} f(h_{pq})$, which *only works when* there are *no* chiral breaking terms in the supersymmetry constraints (otherwise it mixes fermion terms of different order).

To proceed, the *complete* set of coupled first-order partial differential constraint equations must be investigated. In [70], this is present in the form of a set of coupled first-order partial differential equations for a diagonal Bianchi IX model. In this context, the most general expression for Ψ is

$$\Psi = \mathsf{A}(h_{mn}) + \mathsf{C}_{pq}(h_{mn})\psi^{pA}\psi^{q}_{A} + \mathsf{V}^{pqr}(h_{mn})n_{AA'}e_{pB}{}^{A'}\psi^{A}_{q}\psi^{B}_{r}$$
$$+\Psi_4 + \mathsf{F}(h_{mn})\prod_{i=1}^{3}\psi^{iA}\psi_{iA}\ , \tag{5.209}$$

where C_{pq} is symmetric and V^{pqr} antisymmetric in the last two indices, C_{pq} and V^{pqr} provide 6 and 9 degrees of freedom, respectively, and

$$\begin{aligned}\Psi_4 = {} & \mathsf{E}_{1122}\psi^{1A}\psi_{1A}\psi^{2B}\psi_{2B} + \mathsf{E}_{1133}\psi^{1A}\psi_{1A}\psi^{3B}\psi_{3B} \\ & +\mathsf{E}_{2233}\psi^{2A}\psi_{2A}\psi^{3B}\psi_{3B} \end{aligned} \tag{5.210}$$

$$\begin{aligned} & +\mathsf{E}_{1123}\psi^{1A}\psi_{1A}\psi^{2B}\psi_{3B} + \mathsf{E}_{2213}\psi^{2A}\psi_{2A}\psi^{1B}\psi_{3B} \\ & +\mathsf{E}_{3312}\psi^{3A}\psi_{3A}\psi^{1B}\psi_{2B} \end{aligned} \tag{5.211}$$

$$\begin{aligned} & +\mathsf{F}^{p}{}_{1233}e_{pB}{}^{A'}n_{AA'}\psi^{1A}\psi^{2B}\psi^{3C}\psi^{3}{}_{C} \\ & +\mathsf{F}^{p}{}_{1323}e_{pB}{}^{A'}n_{AA'}\psi^{1A}\psi^{3B}\psi^{2C}\psi^{2}{}_{C} \\ & +\mathsf{F}^{p}{}_{2311}e_{pB}{}^{A'}n_{AA'}\psi^{2A}\psi^{3B}\psi^{1C}\psi^{1}{}_{C} \ , \end{aligned} \tag{5.212}$$

with the E and V providing 6 and 9 degrees of freedom, respectively. This constitutes the most general solution to the Lorentz constraints. A sequence of seven pillars is required to hold up this edifice:

1. For $\overline{\mathcal{S}}_{A'}\Psi = 0$ at the one-fermion level, it follows that

$$\varepsilon^{pqr}e_{pAA'}\omega_q{}^{A}{}_{B}\psi_r^B\mathsf{A} + \hbar e_{qAA'}\psi_p^A\frac{\partial\mathsf{A}}{\partial h_{pq}} = 0 \ , \tag{5.213}$$

where the off-diagonal equations are eliminated, since this implies $0 = 0$ when $q \neq p$. For all ψ_r^B, and after multiplying by $e_l{}^{BA'}$,

$$\mathrm{i}\left(h_{ql}n^{A}{}_{A'}e^{pBA'}\omega_{pAB} - n^{A}{}_{A'}e_q{}^{AB'}\omega_{lAB}\right)\mathsf{A} - \hbar h_{pl}h_{qs}\frac{\partial\mathsf{A}}{\partial h_{ps}} = 0 \ , \tag{5.214}$$

whereupon

$$\hbar\frac{\partial\mathsf{A}}{\partial a_1} + a_1\mathsf{A} = 0 \ , \tag{5.215}$$

$$\hbar\frac{\partial\mathsf{A}}{\partial a_2} + a_2\mathsf{A} = 0 \ , \tag{5.216}$$

$$\hbar\frac{\partial\mathsf{A}}{\partial a_3} + a_3\mathsf{A} = 0 \ , \tag{5.217}$$

i.e.,

$$\mathsf{A} \propto \exp\left[-\frac{1}{2\hbar}\left(a_1^2 + a_2^2 + a_3^2\right)\right] . \tag{5.218}$$

2. For $\mathcal{S}_A\Psi = 0$ at the one-fermion level, one has

$$
\begin{aligned}
0 &= 2\mathsf{C}_{pq}h^{qv}\omega^p{}_{AD} - 2\mathsf{V}^{tuv}\omega_{uA}{}^B n_{BC'}e_{tD}{}^{C'} \\
&\quad + 2\hbar h^{tv}\frac{\partial \mathsf{C}_{pt}}{\partial h_{qr}} D_{D_q^{A'p}} e_{rAA'} - 2\hbar \frac{\partial \mathsf{V}^{tuv}}{\partial h_{pq}} D^{BA'}{}_{uq} e_{pAA'} n_{BC'} e_{tD}{}^{C'} \\
&\quad + \hbar \mathsf{V}^{tuv} D^{BA'}{}_{uq} n_{AA'} e^q{}_{BC'} e_{tD}{}^{C'} + \mathrm{i}\hbar h_{ut}\mathsf{V}^{tuv}\varepsilon_{AD} \; .
\end{aligned}
\tag{5.219}
$$

a. Contracting the indices A and D with ε^{AD}, using $v = 1, 2, 3$, respectively, three equations are retrieved which are just given by cyclic permutations of a_1, a_2, a_3.
b. Multiplying (5.219) by $e_x{}^{AD'} n^D{}_{D'}$, it follows that

$$
\begin{aligned}
0 &= \hbar\left(2h_{xr}\frac{\partial \mathsf{C}_{pw}}{\partial h_{rp}} - h_{rs}\frac{\partial \mathsf{C}_{xw}}{\partial h_{rs}}\right) + 2\mathsf{C}_{pw}\omega^p{}_{AD} n^D{}_{D'} e_x{}^{AD'} \\
&\quad -\hbar \frac{\partial \mathsf{V}^{tuv}}{\partial h_{pq}} h_{qu} h_{vw}\varepsilon_{txp} + \frac{\hbar}{2}\frac{\partial \mathsf{V}^{tuv}}{\partial h_{pq}} h_{pq} h_{vw}\varepsilon_{txu} \\
&\quad + \mathrm{i} h_{vw}\mathsf{V}^{tuv}\varepsilon_{txy}\omega_{uAB} n^B{}_{D'} e^{yAD'} \\
&\quad + \frac{3}{4}\hbar h_{vw}\mathsf{V}^{tuv}\varepsilon_{txu} \; .
\end{aligned}
\tag{5.220}
$$

c. For C_{xw}, $x \neq w$, six equations are obtained:

$$
\begin{aligned}
0 &= \frac{a_2^2}{4}\left[\hbar\left(a_3\frac{\partial}{\partial a_3} - a_2\frac{\partial}{\partial a_2} - a_1\frac{\partial}{\partial a_1}\right) + (a_2^2 + a_1^2 - a_3^2) - 3\hbar\right]\mathsf{V}^{232} \\
&\quad + \frac{\mathrm{i}}{2}\left[\hbar\left(a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2} - a_3\frac{\partial}{\partial a_3}\right) + (a_3^2 + a_2^2 - a_1^2)\right]\mathsf{C}_{12} \; ,
\end{aligned}
\tag{5.221}
$$

$$
\begin{aligned}
0 &= \frac{a_1^2}{4}\left[\hbar\left(a_3\frac{\partial}{\partial a_3} - a_2\frac{\partial}{\partial a_2} - a_1\frac{\partial}{\partial a_1}\right) + (a_1^2 + a_2^2 - a_3^2) - 3\hbar\right]\mathsf{V}^{113} \\
&\quad + \frac{\mathrm{i}}{2}\left[\hbar(a_2\frac{\partial}{\partial a_2} - a_1\frac{\partial}{\partial a_1} - a_3\frac{\partial}{\partial a_3}) + (a_3^2 + a_1^2 - a_2^2)\right]\mathsf{C}_{12} \; .
\end{aligned}
\tag{5.222}
$$

The other four equations are also cyclic permutations of the above on a_1, a_2, a_3.

3. For $\overline{\mathcal{S}}_{A'}\Psi = 0$ at three-fermion level, the method in leads to

$$
\mathsf{C}_{12} \propto \exp\left[-\frac{1}{2\hbar}(a_1^2 + a_2^2 + a_3^2)\right] \; ,
$$

$$
\mathsf{C}_{13} \propto \exp\left[-\frac{1}{2\hbar}(a_1^2 + a_2^2 + a_3^2)\right] \; ,
$$

$$
\begin{aligned}
\mathsf{C}_{23} &\propto \exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{112} &\propto \frac{1}{a_1^3 a_2 a_3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{113} &\propto \frac{1}{a_1^3 a_2 a_3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{212} &\propto \frac{1}{a_1 a_2^3 a_3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{223} &\propto \frac{1}{a_1 a_2^3 a_3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{313} &\propto \frac{1}{a_1 a_2 a_3^3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] ,\\
\mathsf{V}^{323} &\propto \frac{1}{a_1 a_2 a_3^3}\exp\left[-\frac{1}{2\hbar}(a_1^2+a_2^2+a_3^2)\right] . \qquad (5.223)
\end{aligned}
$$

However, these expressions are *not* solutions of the equations retrieved from $\mathcal{S}$ at the one-fermion level, e.g., they are found *not* to satisfy (5.221) and (5.222). The solutions require $\mathsf{C}^0_{12}, \ldots, \mathsf{V}^0_{323} = 0$. These amplitudes are not the dynamical degrees of freedom of the theory, which are contained in the remaining coefficients.

4. Let us therefore investigate the sector of C_{xx} , i.e., $x = w$ and no sum in x. This is where the physical degrees of freedom will be identified along with a relation between the functionals f, g above and the expansion (5.209) of, allowing novel solutions to be extracted. Hence, with $\mathcal{S}_A \Psi = 0$ at the one-fermion level and $\overline{\mathcal{S}}_{A'} \Psi = 0$ at the three-fermion level, from the diagonal elements such as C_{11}, it follows that

$$
\begin{aligned}
0 = a_1^2 &\left[\frac{\hbar}{2}\left(a_3\frac{\partial}{\partial a_3} - a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2}\right) + \left(a_1^2 + a_2^2 - a_3^2\right) - \frac{3}{2}\hbar\right]\mathsf{V}^{231}\\
+a_1^2 &\left[\frac{\hbar}{2}\left(a_3\frac{\partial}{\partial a_3} + a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2}\right) - \left(a_1^2 + a_3^2 - a_2^2\right) + \frac{3}{2}\hbar\right]\mathsf{V}^{321}\\
+\mathrm{i}\hbar &\left(a_1\frac{\partial}{\partial a_1} - a_3\frac{\partial}{\partial a_3} - a_2\frac{\partial}{\partial a_2}\right) C_{11}\\
+\mathrm{i} &\left(a_3^2 + a_2^2 - a_1^2\right)\mathsf{C}_{11} , \qquad (5.224)
\end{aligned}
$$

$$
\begin{aligned}
0 = a_2^2 &\left[\frac{\hbar}{2}\left(-a_3\frac{\partial}{\partial a_3} + a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2}\right) - \left(a_1^2 + a_2^2 - a_3^2\right) + \frac{3}{2}\hbar\right]\mathsf{V}^{132} \\
&+ a_2^2\left[\frac{\hbar}{2}\left(-a_3\frac{\partial}{\partial a_3} + a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2}\right) + \left(a_2^2 + a_3^2 - a_1^2\right) - \frac{3}{2}\hbar\right]\mathsf{V}^{312} \\
&+ \mathrm{i}\hbar\left(a_2\frac{\partial}{\partial a_2} - a_3\frac{\partial}{\partial a_3} - a_1\frac{\partial}{\partial a_1}\right)C_{22} \\
&+ \mathrm{i}\left(a_3^2 + a_1^2 - a_2^2\right)\mathsf{C}_{22}\ , \qquad (5.225)
\end{aligned}
$$

$$
\begin{aligned}
0 = a_3^2 &\left[\frac{\hbar}{2}\left(a_3\frac{\partial}{\partial a_3} - a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2}\right) - \left(a_3^2 + a_2^2 - a_1^2\right) + \frac{3}{2}\hbar\right]\mathsf{V}^{213} \\
&+ a_1^2\left[\frac{\hbar}{2}\left(-a_3\frac{\partial}{\partial a_3} - a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2}\right) + \left(a_1^2 + a_3^2 - a_2^2\right) - \frac{3}{2}\hbar\right]\mathsf{V}^{123} \\
&+ \mathrm{i}\hbar\left(-a_1\frac{\partial}{\partial a_1} - a_3\frac{\partial}{\partial a_3} + a_2\frac{\partial}{\partial a_2}\right)C_{33} \\
&+ \mathrm{i}\left(a_1^2 + a_2^2 - a_3^2\right)\mathsf{C}_{33}\ . \qquad (5.226)
\end{aligned}
$$

As above, some intermediate steps are required:

a. Multiplying by ${e_1}^{BA'}$, then

$$\hbar a_1\frac{\partial\mathsf{C}_{22}}{\partial a_1} + a_1^2\mathsf{C}_{22} - \frac{\mathrm{i}}{2}a_2^2\left(\hbar a_2\frac{\partial}{\partial a_2} + \hbar + a_2^2\right)\mathsf{V}^{321} = 0\ , \qquad (5.227)$$

with the cyclic permutations

$$\hbar a_2\frac{\partial\mathsf{C}_{33}}{\partial a_2} + a_2^2\mathsf{C}_{33} - \frac{\mathrm{i}}{2}a_3^2\left(\hbar a_3\frac{\partial}{\partial a_3} + \hbar + a_3^2\right)\mathsf{V}^{132} = 0\ , \qquad (5.228)$$

$$\hbar a_3\frac{\partial\mathsf{C}_{11}}{\partial a_3} + a_3^2\mathsf{C}_{11} - \frac{\mathrm{i}}{2}a_1^2\left(\hbar a_1\frac{\partial}{\partial a_1} + \hbar + a_1^2\right)\mathsf{V}^{213} = 0\ . \qquad (5.229)$$

b. Similarly, multiplying the equation obtained from $\psi_1\psi_3\psi_3$ by ${e_1}^{BA'}$,

$$\hbar a_2\frac{\partial\mathsf{C}_{11}}{\partial a_2} + a_2^2\mathsf{C}_{11} + \frac{\mathrm{i}}{2}a_1^2\left(\hbar a_1\frac{\partial}{\partial a_1} + \hbar + a_1^2\right)\mathsf{V}^{312} = 0\ , \qquad (5.230)$$

$$\hbar a_3\frac{\partial\mathsf{C}_{22}}{\partial a_3} + a_3^2\mathsf{C}_{22} + \frac{\mathrm{i}}{2}a_2^2\left(\hbar a_2\frac{\partial}{\partial a_2} + \hbar + a_2^2\right)\mathsf{V}^{123} = 0\ , \qquad (5.231)$$

$$\hbar a_1\frac{\partial\mathsf{C}_{33}}{\partial a_1} + a_1^2\mathsf{C}_{33} + \frac{\mathrm{i}}{2}a_3^2\left(\hbar a_3\frac{\partial}{\partial a_3} + \hbar + a_3^2\right)\mathsf{V}^{231} = 0\ . \qquad (5.232)$$

c. Three more equations are obtained by multiplying by $n^{EA'}$ and taking cyclic permutations:

$$\left(\hbar a_2 \frac{\partial}{\partial a_2} + a_2^2 + 3\hbar\right) \mathsf{V}^{213} + \left(\hbar a_3 \frac{\partial}{\partial a_3} + a_3^2 + 3\hbar\right) \mathsf{V}^{312} = 0 \,, \quad (5.233)$$

$$\left(\hbar a_1 \frac{\partial}{\partial a_1} + a_1^2 + 3\hbar\right) \mathsf{V}^{123} + \left(\hbar a_3 \frac{\partial}{\partial a_3} + a_3^2 + 3\hbar\right) \mathsf{V}^{321} = 0 \,, \quad (5.234)$$

$$\left(\hbar a_1 \frac{\partial}{\partial a_1} + a_1^2 + 3\hbar\right) \mathsf{V}^{132} + \left(\hbar a_2 \frac{\partial}{\partial a_2} + a_2^2 + 3\hbar\right) \mathsf{V}^{231} = 0 \,. \quad (5.235)$$

5. Finally, assuming that the coefficients have the form

$$\mathsf{C}_{11} = \left(\mathsf{C}_{(0)11} + \hbar \mathsf{C}_{(1)11} + \hbar^2 \mathsf{C}_{(2)11} + \cdots\right) \mathrm{e}^{-I/\hbar} \,, \quad \text{etc.} \,,$$
$$\mathsf{V}^{123} = \left(\mathsf{V}^{123}_{(0)} + \hbar \mathsf{V}^{123}_{(1)} + \hbar^2 \mathsf{V}^{123}_{(2)} + \cdots\right) \mathrm{e}^{-I/\hbar} \,, \quad \text{etc.} \,, \quad (5.236)$$

where I is a classical Euclidean action, yields the Hamilton–Jacobi equations.

6. In fact, from

$$0 = \mathrm{i}\left(-a_1 \frac{\partial I}{\partial a_1} + a_2 \frac{\partial I}{\partial a_2} + a_3 \frac{\partial I}{\partial a_3} - a_1^2 - a_2^2 - a_3^2\right) \mathsf{C}_{(0)11}$$
$$+ \frac{a_1^2}{2}\left(a_2 \frac{\partial I}{\partial a_2} + a_1 \frac{\partial I}{\partial a_1} - a_3 \frac{\partial I}{\partial a_3} + a_1^2 + a_2^2 - a_3^2\right) \mathsf{V}^{231}_{(0)}$$
$$+ \frac{a_1^2}{2}\left(a_2 \frac{\partial I}{\partial a_2} - a_1 \frac{\partial I}{\partial a_1} - a_3 \frac{\partial I}{\partial a_3} - a_1^2 + a_2^2 - a_3^2\right) \mathsf{V}^{321}_{(0)} \,, \quad (5.237)$$

$$a_3\left(-\frac{\partial I}{\partial a_3} + a_3\right) C_{(0)11} + \frac{\mathrm{i}}{2} a_1^3 \left(-\frac{\partial I}{\partial a_1} + a_1\right) V^{231}_{(0)} = 0 \,, \quad (5.238)$$

$$a_2\left(-\frac{\partial I}{\partial a_2} + a_2\right) \mathsf{C}_{(0)11} - \frac{\mathrm{i}}{2} a_1^3 \left(-\frac{\partial I}{\partial a_1} + a_1\right) \mathsf{V}^{321}_{(0)} = 0 \,, \quad (5.239)$$

$$a_2\left(-\frac{\partial I}{\partial a_2} + a_2\right) \mathsf{V}^{231}_{(0)} + a_3 \left(-\frac{\partial I}{\partial a_3} + a_3\right) \mathsf{V}^{321}_{(0)} = 0 \,, \quad (5.240)$$

there follows, e.g., an equation homogeneous in $\mathsf{C}_{(0)11}$, whose non-trivial solution requires

$$0 = a_1^2 \left(\frac{\partial I}{\partial a_1}\right)^2 + a_2^2 \left(\frac{\partial I}{\partial a_2}\right)^2 + a_3^2 \left(\frac{\partial I}{\partial a_3}\right)^2 - 2a_1a_2 \left(\frac{\partial I}{\partial a_1}\right)\left(\frac{\partial I}{\partial a_2}\right)$$
$$-2a_1a_3 \left(\frac{\partial I}{\partial a_1}\right)\left(\frac{\partial I}{\partial a_3}\right) - 2a_2a_3 \left(\frac{\partial I}{\partial a_2}\right)\left(\frac{\partial I}{\partial a_3}\right)$$
$$-a_1^4 - a_2^4 - a_3^4 + 2a_1^2a_2^2 + 2a_1^2a_3^2 + 2a_2^2a_3^2 \ . \qquad (5.241)$$

7. Since I has the general form,

$$I = \alpha a_1^2 + \beta a_2^2 + \gamma a_3^2 + \mu a_1a_2 + \nu a_2a_3 + \lambda a_1a_3 \ , \qquad (5.242)$$

substituting it into the above Hamilton–Jacobi type equation gives

$$0 = (4\alpha^2 - 1)a_1^4 + (4\beta^2 - 1)a_2^4 + (4\gamma^2 - 1)a_3^4$$
$$+(2 - 8\alpha\beta)a_1^2a_2^2 + (2 - 8\gamma\alpha)\, a_1^2a_3^2 + (2 - 8\beta\gamma)a_2^2a_3^2$$
$$-4(2\alpha\nu + \mu\lambda)a_1^2a_2a_3 - 4(2\beta\gamma + \nu\mu)a_1a_2^2a_3$$
$$-4(2\gamma\mu + \nu\lambda)a_1a_2a_3^2 \ . \qquad (5.243)$$

The most general solutions are [70]

$$\pm\, I = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right) \ ,$$
$$\pm I = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right) - a_1a_2 - a_1a_3 - a_2a_3 \ ,$$
$$\pm I = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right) + a_1a_2 + a_1a_3 - a_2a_3 \ ,$$
$$\pm I = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right) + a_1a_2 - a_2a_3 + a_2a_3 \ ,$$
$$\pm I = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right) - a_1a_2 + a_1a_3 + a_2a_3 \ . \qquad (5.244)$$

The first is the wormhole action and the second is the Hartle–Hawking action, existing in the same fermion sector.

Note 5.12 The analysis and construction above [70], although clearly extending from the methodology in [68, 69], can be employed even if there are chiral breaking terms in the supersymmetry constraints in pure $N = 1$ supergravity, i.e., when a cosmological constant is present (see Exercise 5.6). The approach in [68, 69] *cannot* proceed beyond chiral breaking terms, which will not preserve the number of fermions and gives mixing of different levels of fermions, e.g., when supermatter is present.

Summary and Review. With the aim of both assisting and guiding the reader on his or her progression through the book, here is a list of items for assessing the content of Chap. 5:

1. How precisely can we implement a FRW SQC (minisuperspace) from $N = 1$ SUGRA and what are the corresponding quantum solutions (Sects. 5.1.1 and 5.1.2)?
2. What is the pertinent feature that a cosmological constant brings into a FRW SQC (minisuperspace) obtained from $N = 1$ SUGRA (Sect. 5.1.3)?
3. How can scalar (super)matter be included and what can be done when there is a potential for the scalar field (Sect. 5.1.4)?
4. How have vector supermultiplets been dealt with (Sect. 5.1.5)?
5. Why are Bianchi class A models relevant here, and what are the possible approaches for implementing a Bianchi class A minisuperspace from $N = 1$ SUGRA (Sects. 5.2, 5.2.1, and 5.2.2)?
6. How did the content of the quantum SQC Bianchi state spectrum 'evolve'? What are the main ingredients for the 'solution' (Sect. 5.2.3)?
7. Why is the case of Bianchi models with a cosmological constant still an issue (Sect. 5.2.3)?

Problems

5.1 Evaluating $^3D_i\varepsilon_A$ for $\delta\psi_i^A$

Show that the second line in (5.6) corresponds to evaluating $^3D_i\varepsilon_A$.

5.2 Using ψ_0^A or ρ^A

Using ψ_0^A or

$$\rho^A \equiv \frac{\mathrm{i}(\mathsf{k}\varsigma)^{1/2}}{2\sqrt{6}a^{1/2}}\psi^A{}_0 + \frac{\mathrm{i}\varsigma\mathcal{N}}{12\mathsf{k}a^2}n^{AA'}\overline{\psi}_{A'} ,$$

investigate the supersymmetry and Hamiltonian constraints for the case of a FRW model *without* any matter.

5.3 On Conserved Currents

Discuss whether conserved currents can be retrieved for FRW supersymmetric minisuperspaces with a scalar supermultiplet and $\mathsf{P} = \check{\mathrm{D}}_\phi\mathsf{P} = 0$. Choose $\mathsf{C} = 0$ in the SUSY wave function (5.73).

5.4 Supersymmetry Invariance for a FRW Model with Vector Fields

Analyse and discuss the choice of field variables for the FRW minisuperspace with vector gauge fields (with no scalar fields and corresponding fermionic partners), regarding SUSY transformations and their possible invariance.

5.5 Alternative Homogeneity Condition and Quantum States for SUSY Bianchi IX

Use the *alternative* homogeneity condition

$$\frac{\partial \psi^{Aa}}{\partial x^i} = -\mathrm{i}\omega^p{}_i \mathcal{O}_p{}^b \left(\frac{1}{\sqrt{2}} \sigma_b{}^{AA'} \delta_{BA'} \psi^{B\,a} + \mathrm{i}\varepsilon^{abc} \psi^{Ac} \right)$$

for the gravitinos, rather than $\partial \psi^{Aa}/\partial x^i = 0$, to solve the quantum constraints for a Bianchi IX model (see [66]).

5.6 Quantum States for SUSY Bianchi IX with a Cosmological Constant

Investigate why, when a cosmological constant $\Lambda < 0$ was added, it led to the undesirable situation that *no* physical states but the trivial one $\Psi = 0$ were found [33, 71].

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Chapter 6
Cosmologies with (Hidden) $N = 2$ Supersymmetry

In this chapter we discuss a rather different methodology[1] for retrieving SQC models: *not directly* from $N = 1$ SUGRA (and therefore involving a dimensional reduction towards an $N = 4$ SUSY minisuperspace), but instead in an opposite direction, starting from purely bosonic configurations (e.g., general relativity and the bosonic sectors of string theories) and then building (in a consistent manner!) a minisuperspace *with* $N = 2$ SUSY, which is the simplest case [1–8].

Of course, the reader may be wondering about the possibility of constructing instead $N = 4$ SUSY minisuperspaces by this route, since this is what was obtained directly from $N = 1$ SUGRA in Chap. 5. However, this is still an *open question* for research. The question is whether we can establish a coherent connection between the methods described here, eventually leading to (new?) $N = 4$ minisuperspaces and relating them to those retrieved directly from $N = 1$ SUGRA (see Chap. 5). Note, however, that a few encouraging attempts can be found in the literature [9–14] (see also Chap. 8).

Let us now proceed with the specific details of $N = 2$ SUSY minisuperspaces built from bosonic models, first in a general relativistic setting in Sect. 6.1, and then describe in Sect. 6.2 how a string theory background changes some of those features.

6.1 Minisuperspaces from General Relativity

In this section we focus on retrieving specific quantized versions for homogeneous cosmologies by coupling additional fermionic degrees of freedom to purely gravitational systems within a general relativistic background. The relevant feature is that the coupled minisuperspace system *acquires* SUSY.

In order to obtain a thorough perspective, we will recall here a few elements regarding diagonal minisuperspace Bianchi models (see Sects. 2.3.1 and 2.4), concerning their Hamiltonian formulation. The 4-metric of a spatially homogeneous spacetime can be written in the form

[1] Whose formal expressions bear interesting similarities with those in Chap. 5.

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$$ds^2 = -\mathcal{N}^2(t)dt^2 + h_{ij}(t)\omega^i\omega^j \ , \tag{6.1}$$

where ω^i constitutes a basis of one-forms, $\mathcal{N}(t)$ is the lapse function, and h_{ij} the 3-metric in the basis ω^i, depending only on time. There are at most 6 independent elements of $h_{ij}(t)$, spanning the space of all allowed 3-metrics, i.e., the minisuperspace. For the case where $h_{ij}(t)$ is diagonal, it can then be parametrized by three parameters α, β_+, and β_-, or β^1, β^2, and β^3 via (see Sect. 2.3.1, where the specific Ω time is employed)

$$h_{ij}(t) = \frac{1}{6\pi}e^{2\alpha(t)}\left[e^{2\beta}(t)\right]_{ij} = \frac{1}{6\pi}e^{2\beta^i}\delta_{ij} \ , \tag{6.2}$$

where $\beta(t)$ is the diagonal traceless matrix specified by

$$\beta_{ij}(t) \equiv \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$$

(see Misner and Ryan's parametrization).[2] Hence, the minisuperspace is spanned by just three coordinates. Note the following points (see (2.67) and (2.68), and Sect. 2.8):

- A dot denotes differentiation with respect to a suitable parameter ζ, playing the role of time.
- q^X is the chosen parametrization (see Sect. 2.8 for the minisuperspace notation) of the independent elements of $h_{ij}(t)$, i.e, $(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$.
- p_Y are the corresponding canonically conjugate momenta.
- $H(q, \mathsf{p}) \sim \mathcal{N}\mathcal{H}$ is the Hamiltonian. The condition $\mathcal{H} = 0$ expresses the reparametrization invariance of the time parameter.
- Therefore, choosing,[3] e.g.,

$$d\zeta = \sqrt{\frac{3\pi}{2}}e^{-3\alpha}\mathcal{N}(t)dt = \sqrt{\frac{3\pi}{2}}e^{-(\beta^1+\beta^2+\beta^3)}\mathcal{N}(t)dt \ ,$$

---

[2] For example, for the expression in (6.2), we need (see footnote 15 in Sect. 2.3.1)

$$\beta_1 \equiv \beta^0 - 2\beta^+ \ , \tag{6.3}$$
$$\beta_2 \equiv \beta^0 + \beta^+ + \sqrt{3}\beta^- \ , \tag{6.4}$$
$$\beta_3 \equiv \beta^0 + \beta^+ - \sqrt{3}\beta^- \ , \tag{6.5}$$

and of course,

$$\beta^0 \equiv \pm\Omega = \pm\ln a = \pm\alpha \ , \tag{6.6}$$

for the Misner–Ryan time coordinate choice(s) [15].

[3] Note that this feature seems to influence the form of the quantum supersymmetry constraints, as will be made clear in subsequent sections.

H takes the form (see Sects. 2.3.1 and 2.3.2)

$$\begin{aligned} H \longleftarrow \mathcal{H} &= \frac{1}{2}(-\mathsf{p}_\alpha^2 + \mathsf{p}_+^2 + \mathsf{p}_-^2) + U^{(0)}(\alpha, \beta_+, \beta_-) \\ &= \frac{3}{2}\Big[(\mathsf{p}_1)^2 + (\mathsf{p}_2)^2 + (\mathsf{p}_3)^2 - 2\mathsf{p}_1\mathsf{p}_2 - 2\mathsf{p}_1\mathsf{p}_3 - 2\mathsf{p}_2\mathsf{p}_3\Big] \\ &\quad + U^{(0)}(\beta^1, \beta^2, \beta^3) \ , \end{aligned} \tag{6.7}$$

with the potential

$$U^{(0)} \equiv -12\pi^2 h^{(3)}R \ , \tag{6.8}$$

where h is the determinant and ${}^{(3)}R$ the scalar curvature of the 3-metric.

- Such geometrodynamics of the Bianchi configuration (see Sect. 2.3) involves a non-definite metric $\mathcal{G}^{(0)}_{XY}$ in minisuperspace (see (2.25) in Sect. 2.2.1 and Exercise 2.6), whose line element in the parametrization by α, β_+, β_- or β^1,β^2, β^3 may be written in the form

$$\mathsf{ds}^2 = \mathcal{G}^{(0)}_{XY}\mathrm{d}q^X\mathrm{d}q^Y \ , \tag{6.9}$$

with $\mathcal{G}^{(0)}_{XY} = \mathrm{diag}(-1, 1, 1)$ or $\mathcal{G}^{(0)}_{XY} = -(1-\delta_{XY})/6$, respectively. Reparametrization invariance with respect to ζ, using $\mathrm{d}\zeta = \mathrm{e}^{2\tilde{\Omega}}\mathrm{d}\zeta^{(0)}$, means that the metric in minisuperspace is fixed only up to an arbitrary conformal factor,[4] here written as $\exp\big(2\tilde{\Omega}(q)\big)$:

$$\mathcal{G}_{XY} = \mathrm{e}^{2\tilde{\Omega}(q)}\mathcal{G}^{(0)}_{XY} \ . \tag{6.10}$$

The inverse of this conformal factor then appears in the potential of (6.7), i.e.,

$$U(q) = \mathrm{e}^{-2\tilde{\Omega}(q)}U^{(0)}(q) \ . \tag{6.11}$$

- Meanwhile, continuing along our route of exploration, let us indicate some potentials for the Bianchi cosmologies:

$$\text{Type I}: \qquad \mathrm{U}^{(0)}_{\mathrm{I}} \equiv 0 \ , \tag{6.12}$$

$$\text{Type II}: \qquad \mathrm{U}^{(0)}_{\mathrm{II}} \equiv \frac{1}{6}\mathrm{e}^{4\alpha}\mathrm{e}^{-8\beta_+} = \frac{1}{6}\mathrm{e}^{4\beta^3} \ , \tag{6.13}$$

$$\text{Type VII}: \quad \mathrm{U}^{(0)}_{\mathrm{VII}} \equiv \frac{1}{3}\mathrm{e}^{4\alpha}\mathrm{e}^{4\beta_+}\Big[\cosh(4\sqrt{3}\beta_-) - 1\Big] = \frac{1}{6}\left(\mathrm{e}^{2\beta^1} - \mathrm{e}^{2\beta^2}\right)^2 \ , \tag{6.14}$$

[4] Ω is employed for time coordinate in the ADM parametrization, whereas $\tilde{\Omega}$ is the conformal factor function.

$$\text{Type VIII}: \quad \mathrm{U}^{(0)}_{\mathrm{VIII}} \equiv \frac{1}{6} e^{4\alpha} \Big\{ 2e^{4\beta_+} \Big[\cosh(4\sqrt{3}\beta_-) - 1 \Big] + e^{-8\beta_+} + 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) \Big\}$$
$$= \frac{1}{6} \Big[e^{4\beta^1} + e^{4\beta^2} + e^{4\beta^3} + 2e^{2\beta^3} (e^{2\beta 1} + e^{2\beta^2}) - 2e^{2\beta^1 + 2\beta^2} \Big], \tag{6.15}$$

$$\text{Type IX}: \quad \mathrm{U}^{(0)}_{\mathrm{IX}} \equiv \frac{1}{6} e^{4\alpha} \Big\{ 2e^{4\beta_+} \Big[\cosh(4\sqrt{3}\beta_-) - 1 \Big] + e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) \Big\}$$
$$= \frac{1}{6} \Big(e^{4\beta^1} + e^{4\beta^2} + e^{4\beta^3} - 2e^{2\beta^1 + 2\beta^2} - 2e^{2\beta^1 + 2\beta^3} - 2e^{2\beta^2 + 2\beta^3} \Big). \tag{6.16}$$

- There are a few special cases from the above list, where further symmetries are present. Hence fewer than three degrees of freedom are required:
  - FRW universes are isotropic special cases of the Bianchi types IX, V, and I, where $\mathsf{p}_+ = \mathsf{p}_- = 0$ and $\beta_+ = \beta_- = 0$. For the FRW universe without matter

$$\mathcal{H}_{\mathrm{FRW}} \equiv -\frac{\mathsf{p}_\alpha^2}{2} + U^{(0)}_{\mathrm{FRW}}(\alpha) \,,$$
$$U^{(0)}_{\mathrm{FRW}} \equiv -\frac{k}{2} e^{4\alpha} \,. \tag{6.17}$$

  - The Kantowski–Sachs model is *not* a Bianchi type. The spacetime metric is (usually) found from the basis

$$\omega^3 = dr \,, \quad \omega^1 = d\theta \,, \quad \omega^2 = \sin\theta d\varphi \,, \tag{6.18}$$
$$\beta(t) \equiv \mathrm{diag}(\beta_+, \beta_+, -2\beta_+) \,, \tag{6.19}$$

  and then

$$\mathcal{H}_{\mathrm{KS}} = -\frac{\mathsf{p}_\alpha^2}{2} + \frac{\mathsf{p}_+^2}{2} + U^{(0)}_{\mathrm{KS}}(\alpha, \beta_+)$$
$$= -3\mathsf{p}_1\mathsf{p}_3 + \frac{3}{2}(\mathsf{p}_3)^2 + U^{(0)}_{\mathrm{KS}}(\beta^1, \beta^3) \,, \tag{6.20}$$
$$U^{(0)}_{\mathrm{KS}} \equiv -\frac{2}{3} e^{4\alpha} e^{-2\beta_+} = -\frac{2}{3} e^{2(\beta^1 + \beta^3)} \,. \tag{6.21}$$

  - In the Taub–NUT space, the Taub space is of Bianchi type IX with a rotational symmetry around one spatial axis (e.g., $\beta_- = 0$, $\mathsf{p}_- = 0$) and

$$\mathcal{H}_{\mathrm{T}} = -\frac{\mathsf{p}_\alpha^2}{2} + \frac{\mathsf{p}_+^2}{2} + U_{\mathrm{T}}^{(0)}(\alpha, \beta_+)$$
$$= -3\mathsf{p}_1\mathsf{p}_2 + \frac{3}{2}(\mathsf{p}_2)^2 + U_{\mathrm{T}}^{(0)}(\beta^1, \beta^3) \,, \tag{6.22}$$
$$U_{\mathrm{T}}^{(0)} \equiv \frac{1}{6}\mathrm{e}^{4\alpha}(\mathrm{e}^{-8\beta_+} - 4\mathrm{e}^{-2\beta_+}) = \frac{1}{6}\left(\mathrm{e}^{4\beta^3} - 4\mathrm{e}^{2\beta^1+2\beta^3}\right) . \tag{6.23}$$

6.1.1 N=2 Supersymmetric Cosmologies

But from *where* will SUSY (regarding (pure) bosonic systems with gravitational degrees of freedom, e.g., in Chap. 2) be either extracted or induced? The fascinating point is that, as we will explain, those models do indeed possess a *hidden* supersymmetry [1, 3, 16, 4, 17].

Note 6.1 There is an important point to keep in mind. As disclosed at the start of this chapter, a dimensional reduction from $(1+3)$-dimensional $N = 1$ SUGRA leads to $N = 4$ SUSY. But the extension we shall describe here only leads to the $N = 2$ type! Can it be considered as a subsymmetry of the larger $N = 4$ supersymmetry obtained (directly) from supergravity? An answer to this question and an *extension* of the bosonic homogeneous models to $N = 4$ SUSY *remains* to be achieved (see Chap. 8).

So how can this be done? For it to be possible, the potentials $U^{(0)}$ in (6.12), (6.13), (6.14), (6.15), and (6.16) must be derivable (literally) from *another* underlying[5] potential, viz., W (see (3.31), (C.81) and Exercise 3.6).

We consider the dynamics of the universe in the minisuperspace potential $U^{(0)}$, henceforth dropping the superscript (0) from, e.g., (6.12), (6.13), (6.14), (6.15), and (6.16), and evolving on a *curved* manifold (configuration space) of arbitrary dimension with minisuperspace metric (see Sect. 2.8 for the notation)

$$\mathsf{ds}^2 = \mathcal{G}_{XY}(q)\mathrm{d}q^X \mathrm{d}q^Y \,. \tag{6.24}$$

SUSY is retrieved when the potential $U(q)$ is derivable from a *superpotential* $\mathsf{W}(q)$ (see, e.g., (3.31)), satisfying

$$U(q) = \frac{1}{2}\mathcal{G}^{XY}(q)\frac{\partial \mathsf{W}(q)}{\partial q^X}\frac{\partial \mathsf{W}(q)}{\partial q^Y} \,, \tag{6.25}$$

and having the *same* symmetries as $\mathcal{H}_{\mathrm{c}}$ (the classical Hamiltonian). This is indeed important:

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[5] At this stage, the reader might find it useful to glance through Sect. 3.3 of Vol. II. In fact, most of what follows here concerns a supersymmetric extension of particle motion in a potential well, i.e., supersymmetric quantum mechanics (SQM).

- The form of the potential W in (6.25) ensures that the Hamiltonian $\mathcal{H}_c$ would be the bosonic component of a supersymmetric Hamiltonian. The symmetry properties must be preserved when quantizing the system.
- Equation (6.25) is the Hamilton–Jacobi equation corresponding to (6.7) and (6.8) in *Euclidean* time. In other words, $\mathsf{W}(q)$ is here a Euclidean action of classical general relativity (see Sect. 2.8.1).
- If a final point (in configuration space) is accessible from the initial point by a classically allowed path, the most likely path followed in the tunnelling process is given by the solutions of the classical Euclidean equations

$$\mathsf{p}_Y = \mathcal{G}_{XY} \frac{\mathrm{d}q^X}{\mathrm{d}\zeta} = \frac{\partial \mathsf{W}}{\partial q^Y} , \tag{6.26}$$

which must be solved under the condition that the path $q(\zeta)$ connects the chosen initial point with the given final point.
- W may become *imaginary*. While in (6.25) this has no immediate consequence, the SUSY extended Hamiltonian then has *unusual* properties.[6]

Determining the Superpotentials W

The superpotentials[7] W for some of the homogeneous models are as follows:

- **Bianchi Type I.** $\mathsf{W}_\mathrm{I}(q)$ should preserve the invariance of $\mathcal{H}_c$ under arbitrary shifts $\delta\alpha$, $\delta\beta_+$, $\delta\beta_-$ or $\delta\beta^1$, $\delta\beta^2$, $\delta\beta^3$, which requires

$$\mathsf{W}_\mathrm{I} \equiv 0 . \tag{6.27}$$

- **Bianchi Type II.** Only the invariance under independent shifts $\delta\beta^1$, $\delta\beta^2$ remains. This restricts the allowed solutions of (6.25) to a function of β^3, which is obtained as

$$\mathsf{W}_\mathrm{II} \equiv \frac{1}{6}\mathrm{e}^{2\alpha - 4\beta_+} = \frac{1}{6}\mathrm{e}^{2\beta^3} . \tag{6.28}$$

- **Bianchi Type IX.** There are three symmetries $\beta^i \leftrightarrow \beta^j (i \neq j)$. These are preserved by

$$\begin{aligned} \mathsf{W}_\mathrm{IX}^{(0)} &\equiv \frac{1}{6}\mathrm{e}^{2\alpha}\left(2\mathrm{e}^{2\beta_+}\cosh 2\sqrt{3}\beta_- + \mathrm{e}^{-4\beta_+}\right) \\ &= \frac{1}{6}\left(\mathrm{e}^{2\beta^1} + \mathrm{e}^{2\beta^2} + \mathrm{e}^{2\beta^3}\right) . \end{aligned} \tag{6.29}$$

---

[6] Perhaps a quick glance at (6.42) may help to illustrate this point.

[7] It is sufficient to examine the case $\widetilde{\Omega}(q) = 0$, where the metric $\mathcal{G}_{XY}\big(= \mathcal{G}_{XY}^{(0)}\big)$ is flat, and $U(q)$ is given by $U^{(0)}(q)$ (see Exercise 6.1).

A further solution exists, which is given by

$$\begin{aligned}\tilde{\mathsf{W}}_{\mathrm{IX}} &= \mathsf{W}^{(0)}_{\mathrm{IX}}\left(\frac{\alpha}{2}, \frac{\beta_+}{2}, \frac{\beta_-}{2}\right) \\ &= \frac{1}{6}\left(\mathrm{e}^{2\beta^1} + \mathrm{e}^{2\beta^2} + \mathrm{e}^{2\beta^3} - 2\mathrm{e}^{\beta^1+\beta^2} - 2\mathrm{e}^{\beta^1+\beta^3} - 2\mathrm{e}^{\beta^2+\beta^3}\right) .\end{aligned} \tag{6.30}$$

- **FRW.** For $\mathcal{G}_{XY} \equiv \{-1\}$, we get

$$\mathsf{W}_{\mathrm{FRW}} \equiv \begin{cases} \frac{1}{2}\mathrm{e}^{2\alpha} , & k = 1 , \\ 0 , & k = 0 , \\ \frac{\mathrm{i}}{2}\mathrm{e}^{2\alpha} , & k = -1 . \end{cases} \tag{6.31}$$

Quantization

The classical Hamiltonian constraint has the form [1, 3, 16, 4, 17]

$$\mathcal{H}_{\mathrm{c}} \equiv \frac{1}{2}\mathcal{G}^{XY}(q)\left(\mathsf{p}_X\mathsf{p}_Y + \frac{\partial \mathsf{W}}{\partial q^X}\frac{\partial \mathsf{W}}{\partial q^Y}\right) . \tag{6.32}$$

We associate it with a *quantum* Hamiltonian $\mathcal{H}$, reducing to $\mathcal{H}_{\mathrm{c}}$ in the *classical* limit $\hbar \to 0$, of the form

$$2\mathcal{H} = \overline{\mathcal{S}}\mathcal{S} + \mathcal{S}\overline{\mathcal{S}} , \tag{6.33}$$

where $\mathcal{S}$ and $\overline{\mathcal{S}}$ are linear operators[8] satisfying

$$\mathcal{S}^2 = 0 = \overline{\mathcal{S}}^2 . \tag{6.34}$$

The operators $\mathcal{S}$ and $\overline{\mathcal{S}}$ have the explicit form (see Sect. 3.3 of Vol. II)

$$\mathcal{S} \equiv \psi^x {e_x}^Y(q)\left(\pi_Y + \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^Y}\right) , \tag{6.35}$$

$$\overline{\mathcal{S}} \equiv \overline{\psi}_x e^{xY}(q)\left(\pi_Y - \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^Y}\right) , \tag{6.36}$$

where ${e_x}^Y(q) \equiv \mathrm{e}^{-\widetilde{\Omega}}\delta_x^Y$ is the minisuperspace vielbein associated with $\mathcal{G}^{XY}(q)$, which satisfies[9]

$${e_x}^Y(q){e_y}^X(q)\eta^{xy} \equiv \mathcal{G}^{XY}(q) , \tag{6.37}$$

[8] They do indeed constitute ($N = 2$) SUSY generators or (local) gauge constraints.

[9] X, Y correspond to minisuperspace variables and we employ x, y for the corresponding variables in the corresponding tangent space.

with η^{xy} the *local* 'Lorentz' metric in the minisuperspace *tangent section*. The ψ^x and their adjoints $\overline{\psi}_x$ are fermionic operators, constituting Grassmannian (odd) partners of the q^X, therefore introducing fermionic (!) minisuperspace degrees of freedom and enlarging the (minisuperspace) configuration space. They satisfy

$$\left\{\psi^x, \psi^y\right\} = 0 = \left\{\overline{\psi}_x, \overline{\psi}_y\right\} \ , \tag{6.38}$$

$$\left\{\psi^x, \overline{\psi}_y\right\} = \delta^x_y \ . \tag{6.39}$$

Moreover, the π_Y are operators

$$\pi_Y \equiv -\mathrm{i}\hbar \frac{\partial}{\partial q^Y} + \mathrm{i}\hbar \omega_Y{}^y{}_z \overline{\psi}_y \psi^z \ , \tag{6.40}$$

where the *minisuperspace spin connections* $\omega_Y{}^y{}_z$ are functions of the minisuperspace q^X, defined by

$$\omega_Y{}^y{}_z = -e^y{}_X e_z{}^X{}_{;Y} = -\omega_{Yz}{}^y \ . \tag{6.41}$$

Note 6.2 Here $e_c{}^\mu{}_{;\nu}$ denotes the minisuperspace (Riemann-like) covariant derivative of the vielbein fields, with the corresponding *minisuperspace affine connection*. It acts only on the minisuperspace (Riemannian) index X and not on the Lorentz index x, vanishing identically if the $\mathcal{G}^{XY}(q)$ are independent of the variables q (see Sect. 3.3 of Vol. II).

The explicit form of the Hamiltonian (6.33) follows from (6.36). In the special case where the metric (6.37) is flat and constant, it takes the form [1, 3, 16, 4, 17]

$$\mathcal{H} = -\frac{\hbar^2}{2}\mathcal{G}^{XY}\frac{\partial}{\partial q^Y}\frac{\partial}{\partial q^X} + \frac{1}{2}\mathcal{G}^{YX}\frac{\partial \mathsf{W}}{\partial q^Y}\frac{\partial \mathsf{W}}{\partial q^X} + \frac{\hbar}{2}e_x{}^Y e_y{}^X \frac{\partial^2 \mathsf{W}}{\partial q^Y \partial q^X}[\overline{\psi}^x, \psi^y] \ . \tag{6.42}$$

It is indeed the last term in (6.42) that makes it difficult to accommodate imaginary or complex W.

Furthermore, the fermion number

$$\mathcal{F} \equiv \overline{\psi}_x \psi^x \tag{6.43}$$

is conserved by $\mathcal{H}$, as $[\mathcal{H}, \mathcal{F}] = 0$, and $[\mathcal{S}, \mathcal{F}] = \mathcal{S}$, $[\overline{\mathcal{S}}, \mathcal{F}] = -\overline{\mathcal{S}}$. Therefore the sectors with fixed fermion numbers $\mathcal{F} = f$ ($0 \leq f \leq \mathbf{D}$, where $\mathbf{D}$ is the dimension of the minisuperspace) can be considered *separately*, i.e., the constraints can be solved *separately* in the several sectors of fixed value of $\mathcal{F}$.

Note 6.3 Before quantum mechanical issues are taken into consideration, we should make the following remarks:

- The Hamiltonian (6.33) commutes with $\mathcal{S}$ and $\overline{\mathcal{S}}$ [1, 3, 16, 4, 17]

$$[\mathcal{H}, \mathcal{S}] = 0 = [\mathcal{H}, \overline{\mathcal{S}}] \;, \tag{6.44}$$

  implying that the theory is invariant under the supersymmetry transformation [1]

$$M \longrightarrow M + [M, \tilde{\varepsilon}\mathcal{S}] + [\overline{\mathcal{S}}\varepsilon, M] \;, \tag{6.45}$$

  where ε and $\tilde{\varepsilon}$ are arbitrary parameters, anticommuting among themselves and with all fermionic variables, and commuting with bosonic variables. In particular,

$$q^Y \to q^Y + \mathrm{i}\psi^Y \tilde{\varepsilon}(t) - \mathrm{i}\varepsilon(t)\overline{\psi}^Y \;. \tag{6.46}$$

- Local SUSY, i.e., invariance under the transformation (6.45), must be required, with *time dependent* $\tilde{\varepsilon}(t)$ and $\varepsilon(t)$. This imposes the constraints $\mathcal{S} = 0, \overline{\mathcal{S}} = 0$ on the quantum state Ψ, i.e.,

$$\mathcal{S}\Psi = 0 = \overline{\mathcal{S}}\Psi \;. \tag{6.47}$$

  These constraints then imply a Wheeler–DeWitt equation.

6.1.2 Empty Matter Sector

In this section we apply the previous methods and tools to the simpler case of Bianchi models without *any* matter [1, 3, 17]. We therefore write the SUSY constraints (6.35), (6.36), and (6.47) as $\mathcal{S}_0$ and $\overline{\mathcal{S}}_0$ for the 'pure' gravitational terms, with

$$\mathcal{S}_0 = \psi^x e_x{}^Y \left(-\mathrm{i}\frac{\partial}{\partial q^Y} + \mathrm{i}\omega_Y{}^y{}_z \overline{\psi}_y \psi^z + \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^Y} \right) \;, \tag{6.48}$$

$$\overline{\mathcal{S}}_0 = \overline{\psi}_x e^{xY} \left(-\mathrm{i}\frac{\partial}{\partial q^Y} + \mathrm{i}\omega_Y{}^y{}_z \overline{\psi}_b \psi^z - \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^Y} \right) \;, \tag{6.49}$$

and choose a conformal gauge of the minisuperspace metric in which the prefactor is $\exp(6\alpha)$.

Note 6.4 The reader may be wondering why we have done this. In fact, it is in preparation for the inclusion of a complex scalar field ϕ, $\overline{\phi}$, when a conformal gauge with $e^{6\alpha+\phi^2}$ will be used to *simplify* the superpotential W terms (see Sect. 6.1.4).

Hence (6.48) and (6.49) become

$$\begin{aligned}\mathcal{S}_0 = \mathrm{i}e^{-3\alpha}\Bigg[\psi^0\left(-\hbar\frac{\partial}{\partial\alpha}+\frac{\partial\mathsf{W}}{\partial\alpha}\right)+\psi^1\left(-\hbar\frac{\partial}{\partial\beta_+}-3\hbar\overline{\psi}_1\psi^0+\frac{\partial\mathsf{W}}{\partial\beta_+}\right)\\ +\psi^2\left(-\hbar\frac{\partial}{\partial\beta_-}-3\hbar\overline{\psi}_2\psi^0+\frac{\partial\mathsf{W}}{\partial\beta_-}\right)\Bigg],\end{aligned} \tag{6.50}$$

$$\begin{aligned}\overline{\mathcal{S}}_0 = \mathrm{i}e^{-3\alpha}\Bigg[-\overline{\psi}_0\left(-\hbar\frac{\partial}{\partial\alpha}-\frac{\partial\mathsf{W}}{\partial\alpha}\right)+\overline{\psi}_1\left(-\hbar\frac{\partial}{\partial\beta_+}-3\hbar\overline{\psi}_0\psi^1-\frac{\partial\mathsf{W}}{\partial\beta_+}\right)\\ +\overline{\psi}_2\left(-\hbar\frac{\partial}{\partial\beta_-}-3\hbar\overline{\psi}_0\psi^2-\frac{\partial\mathsf{W}}{\partial\beta_-}\right)\Bigg].\end{aligned} \tag{6.51}$$

The signs in the $\overline{\psi}_0$ terms in $\overline{\mathcal{S}}_0$ should be noted. They are caused by the Minkowskian nature of the minisuperspace, and make $\overline{\mathcal{S}}_0$ *different* from the adjoint of $\mathcal{S}_0$. In addition, all the three-fermions in (6.51) arise from the *spin connection* in the extended minisuperspace, and therefore have a *geometrical* interpretation [1, 3, 17].

However, the choice of prefactor in $\mathcal{G}_{XY}$ is just a matter of convenience (see Sect. 6.1.4). It is therefore of interest to explore other choices. The simplest is $\mathcal{G}_{XY}(q) \equiv \left(\mathcal{G}^{(0)}_{XY} = \right)\eta_{XY}$. The classical Hamiltonian $\mathcal{H}_c$ is then written as

$$\mathcal{H}_c = \frac{1}{2}\eta^{XY}\left(\mathsf{p}_X\mathsf{p}_Y + \frac{\partial\mathsf{W}}{\partial q^X}\frac{\partial\mathsf{W}}{\partial q^Y}\right). \tag{6.52}$$

The expressions for $\mathcal{S}_0$, $\overline{\mathcal{S}}_0$ are now simpler than in (6.51). No connection terms appear for a flat metric [1, 3, 17]. Hence,

$$\begin{aligned}\mathcal{S}_0 &= \mathrm{i}\Bigg[\psi^0\left(-\hbar\frac{\partial}{\partial\alpha}+\frac{\partial\mathsf{W}}{\partial\alpha}\right)+\psi^1\left(-\hbar\frac{\partial}{\partial\beta_+}+\frac{\partial\mathsf{W}}{\partial\beta_+}\right)+\psi^2\left(-\hbar\frac{\partial}{\partial\beta_-}+\frac{\partial\mathsf{W}}{\partial\beta_-}\right)\Bigg],\\ \overline{\mathcal{S}}_0 &= \mathrm{i}\Bigg[\overline{\psi}_0\left(\hbar\frac{\partial}{\partial\alpha}+\frac{\partial\mathsf{W}}{\partial\alpha}\right)+\overline{\psi}_1\left(-\hbar\frac{\partial}{\partial\beta_+}-\frac{\partial\mathsf{W}}{\partial\beta_+}\right)+\overline{\psi}^2\left(-\hbar\frac{\partial}{\partial\beta_+}-\frac{\partial\mathsf{W}}{\partial\beta_-}\right)\Bigg].\end{aligned} \tag{6.53}$$

Quantum States

The algebra (6.39) for Bianchi models admits an 8D matrix representation (see Chap. 7 of Vol. II), which is equivalent to a representation in terms of three Grassmann variables θ^Y and their derivatives, with $\overline{\psi}^Y = \theta^Y$ and $\psi^Y = \mathcal{G}^{XY}\partial/\partial\theta^Y$. We henceforth adopt this throughout the present chapter. Since $\mathcal{H}$ commutes with the fermion number operator, now in the form $\theta^Y \partial/\partial\theta^Y$, any solution can be decomposed into a 3D Grassmann representation for Ψ, in the form

$$\Psi \equiv \mathsf{A}_+ + \mathsf{B}_Y \theta^Y + \frac{1}{2}\varepsilon_{XYZ}\mathsf{C}^Z \theta^X \theta^Y + \mathsf{A}_- \theta^0 \theta^1 \theta^2 \,, \tag{6.54}$$

with X, Y, Z running over 0, 1, 2 and $\mathsf{A}_\pm$, B_Y, and C^Z functionals of α and $\beta_\pm$.

The allowed physical states are subsequently retrieved from the constraints

$$\mathcal{S}\Psi = 0 \,, \qquad \overline{\mathcal{S}}\Psi = 0 \,. \tag{6.55}$$

The solutions for the functions $\mathcal{A}_\pm$ are, as expected,[10]

$$\mathsf{A}_\pm = \mathrm{q}_\pm \mathrm{e}^{\mp \mathsf{W}} \,, \tag{6.56}$$

where $\mathrm{q}_\pm$ are integration constants. As far as B_X and C_Y are concerned, additional elements are required. From the ansatz

$$\mathsf{B}_X \equiv \frac{\partial \mathsf{f}_+}{\partial q^X} \mathrm{e}^{-\mathsf{W}} \,, \tag{6.57}$$

in (6.55) and (6.54), it leads to the master equation for the auxiliary function f_+, viz.,

$$\Box \mathsf{f}_+ - 2\mathcal{G}^{XY} \frac{\partial \mathsf{W}}{\partial q^X} \frac{\partial \mathsf{f}_+}{\partial q^Y} = 0 \,, \tag{6.58}$$

where $\Box$ is the 3D d'Alembertian in the q^X minisuperspace coordinates with signature $(-++)$ (see Sect. 2.3). In addition, it can be shown [17] that from

$$\mathsf{C}^Y \equiv \mathcal{G}^{YX} \frac{\partial \mathsf{f}_-}{\partial q^X} \mathrm{e}^{\mathsf{W}} \,, \tag{6.59}$$

we obtain the second master equation of the form

$$\Box \mathsf{f}_- + 2\mathcal{G}^{XY} \frac{\partial \mathsf{W}}{\partial q^X} \frac{\partial \mathsf{f}_-}{\partial q^Y} = 0 \,. \tag{6.60}$$

Thus, both (6.58) and (6.60) can be rewritten in the condensed form

$$\Box \mathsf{f}_\pm \mp 2\mathcal{G}^{XY} \frac{\partial \mathsf{W}}{\partial q^X} \frac{\partial \mathsf{f}_\pm}{\partial q^Y} = 0 \,. \tag{6.61}$$

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[10] To see why, refer to Sect. 3.3 of Vol. II.

To solve (6.61), the potentials $U(q^X)$ [or $\mathsf{W}(q^X)$] must be specified. Once the $\mathsf{f}_\pm$ are obtained, the remaining bosonic components that appear in the Grassmann expansion of the wave function (6.54) can be determined as follows, with the additional ansatz

$$\mathsf{f}_\pm(q^X) \equiv \mathsf{w}_\pm(q^X)\mathrm{e}^{\pm \mathsf{W}(q^X)} \, . \tag{6.62}$$

Equation (6.61) becomes

$$\Box \mathsf{w}_\pm = \left[(\nabla \mathsf{W})^2 \mp \Box \mathsf{W}\right] \mathsf{w}_\pm \, , \tag{6.63}$$

whose solutions are

$$\mathsf{w}_\pm(q^X) \equiv \flat_\pm \mathrm{e}^{m_X T^X} \mathrm{e}^{\mp W} \, , \tag{6.64}$$

where $\flat_\pm$ are integration constants, $m_X = (m_1, m_2, m_3)$ is a vector of null measure (i.e., $-m_1^2 + m_2^2 + m_3^2 = 0$), and $T^Y = (-\alpha, \beta_+, \beta_-)$. Thus, the solutions for $\mathsf{f}_\pm$ are given by

$$\mathsf{f}_\pm = \flat_\pm \mathrm{e}^{m_\mu T^\mu} \, . \tag{6.65}$$

Finally, the solutions for all bosonic components of the wave function are[11]

$$\begin{aligned} &\mathsf{A}_\pm = q_\pm \mathrm{e}^{\mp \mathsf{W}} \, , && \\ &\mathsf{B}_0 = -m_{1_+} \mathsf{f}_+ \mathrm{e}^{-\mathsf{W}} \, , && \mathsf{C}^0 = m_{1\_} \mathsf{f}_- \mathrm{e}^{\mathsf{W}} \, , \\ &\mathsf{B}_1 = m_{2_+} \mathsf{f}_+ \mathrm{e}^{-\mathsf{W}} \, , && \mathsf{C}^1 = m_{2\_} \mathsf{f}_- \mathrm{e}^{\mathsf{W}} \, , \\ &\mathsf{B}_2 = m_{3_+} \mathsf{f}_+ \mathrm{e}^{-\mathsf{W}} \, , && \mathsf{C}^2 = m_{3\_} \mathsf{f}_- \mathrm{e}^{\mathsf{W}} \, , \end{aligned} \tag{6.66}$$

where $q_\pm$ and $m_{i_\pm}$ are constants.

Tentative Interpretation of the Solutions

In order to make contact with reality, a physical interpretation for the wave function (6.54), (6.66) must be established.

Note 6.5 Any discussion regarding how to interpret the wave function (i.e., taking $|\Psi|^2$ to induce some probability measure) can only be achieved with respect to a given setup for a measurement. However, referring to the entire universe, there is no possibility of a separate measurement. The universe itself acts as its own measurement operation. And this may bring us to the domain

[11] Note that for Bianchi type I, we have plane waves as $\mathsf{W} = 0$.

of decoherence in quantum cosmology [18–23] (see also Chap. 2 of Vol. II and references therein), and extending it into SQC (see [24–26] and Chap. 8). In fact, we need a framework through which the wave function of the SUSY universe reaches a form in which the classical features of the universe become observable. The semi-classical limit analysis would be relevant here (see [27] and Chap. 4 of Vol. II).

As indicated in Chap. 2 (see also [19]), there is as yet no generally accepted interpretation in quantum cosmology (in contrast with the usual statistical intepretation of quantum mechanics). Therefore, regarding the solution(s) indicated in the previous section, some suggestions for interpretation should be put forward.

Using a Normalized or Unnormalised $|\Psi|^2$

We can obtain $|\Psi|^2$ for the wave function (6.54), by *integrating over* the Grassmann variables θ^X:

$$|\Psi|^2 = \mathsf{A}_+^*\mathsf{A}_+ + \mathsf{B}_0^*\mathsf{B}_0 + \mathsf{B}_1^*\mathsf{B}_1 + \mathsf{B}_2^*\mathsf{B}_2 + \mathsf{C}^{0*}\mathsf{C}^0 + \mathsf{C}^{1*}\mathsf{C}^1 + \mathsf{C}^{2*}\mathsf{C}^2 + \mathsf{A}_-^*\mathsf{A}_- \ . \tag{6.67}$$

In the particular case of *empty* Bianchi models, assuming that the constants appearing in (6.67) are real, we then obtain

$$|\Psi|^2 = \left[q_+^2 + (m_{1_+}^2 + m_{2_+}^2 + m_{3_+}^2)\mathsf{f}_+^2\right] \mathrm{e}^{-2\mathsf{W}} + \left[q_-^2 + (m_{1_-}^2 + m_{2_-}^2 + m_{3_-}^2)\mathsf{f}_-^2\right] \mathrm{e}^{2\mathsf{W}}. \tag{6.68}$$

Requiring only the terms $\mathrm{e}^{-2\mathsf{W}}$ to remain in (6.68) is a possible (boundary condition) procedure. This is equivalent to requiring that Ψ does not diverge (at a fixed α) for large spatial anisotropy ($|\beta_\pm| \to \infty$). Then terms such as A_- must be ruled out, i.e., we have to fix the condition that $\mathrm{q}_- = 0$. Similarly for f_-.

The form for the unnormalized probability density (6.68), for any Bianchi class A cosmological model, is therefore reduced to

$$|\Psi|^2 = \left[q_+^2 + (m_{1_+}^2 + m_{2_+}^2 + m_{3_+}^2)\mathsf{f}_+^2\right] \mathrm{e}^{-2\mathsf{W}} \ . \tag{6.69}$$

The main contribution to this $|\Psi|^2$ will depend on the values of the parameter $\flat_+$ that occurs in the function f_+ and on the range used for the coordinates α and $\beta_\pm$. In particular, it may be possible for the non-bosonic terms in Ψ to be more important than the bosonic terms, i.e., $\mathsf{A}_\pm$.

- Regarding the $|\Psi|^2$ in (6.69), applying it to the diagonal Bianchi IX case [3], it decays very rapidly to zero for large scale factors e^α (as there is no cosmological constant or coupling to a massive scalar field). Moreover, it becomes infinitely broad for $\alpha \to -\infty$, as the cosmological singularity is approached, and narrow for $\alpha \geq 1$).

- For the case of Bianchi type IX to be acceptable, i.e., non-divergent for $\beta_\pm \to \infty$ at fixed α, physical solutions do exist[12]:
  - Equation (6.29) corresponds to a 3-metric as the *inner* boundary of the 4-metric, associated with a wormhole [28]. The associated wave function vanishes rapidly with large volume ($\alpha \to \infty$), i.e., the probability amplitudes for paths of all larger 3-geometries interfere destructively, but constructively for small 3-geometries (on the Planck scale).
  - Equation (6.30) corresponds to spacetimes with 3-metrics, which are regular for small 3-geometries, and the given 3-geometry must be imposed at the *outer* boundary of these spacetimes. This corresponds to a cosmological solution.

Conserved Probability from the SUSY Constraints

Can the success of Dirac's construction of a conserved probability current (which became possible after taking the *square root* of the Klein–Gordon equation) also be achieved in this $N = 2$ SQC? In the context of the current method, it seems that it *cannot* [1, 3].

For the case $\mathcal{G}_{XY}\big(= \mathcal{G}^{(0)}_{XY}\big) = \eta_{XY}$ and W real, with the state Ψ and its adjoint with respect to the fermionic variables, and with the supersymmetry constraints rewritten as [1, 3]

$$(\mathcal{S} + \overline{\mathcal{S}})\Psi = 0 \,, \qquad (\mathcal{S} - \overline{\mathcal{S}})\Psi = 0 \,, \tag{6.70}$$

we introduce the fermions ξ^X, χ^Y by

$$\xi^Y \equiv \psi^Y + \overline{\psi}^Y \,, \qquad \chi^Y \equiv \mathrm{i}(\psi^Y - \overline{\psi}^Y) \,, \tag{6.71}$$

with the properties

$$\begin{aligned} &\left(\xi^0\right)^+ = -\xi^0 \,, \quad \left(\chi^0\right) = -\chi^0 \,, \qquad \left(\xi^i\right)^+ = \xi^i \,, \qquad \left(\chi^i\right) = -\chi^i \,, \quad i = 1, 2, \\ &\{\xi^Y, \xi^X\} = 2\eta^{YX} = \{\chi^Y, \chi^X\} \,, \qquad \{\xi^Y, \chi^X\} = 0 \,. \end{aligned} \tag{6.72}$$

A set of equations is then obtained, which, combined with the α dependence of W, *prevents* the occurrence of conservation laws with a positive probability density.[13] Only if $\mathsf{W} = 0$ (as in Bianchi type I models), does one have conserved currents with a positive density. The reader should investigate Exercise 6.4.

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[12] Concerning the FRW case, see Exercise 6.3.

[13] In fact, $\mathcal{S}\mathsf{A}_+ = 0$ solved for $\partial \mathsf{A}_+/\partial\alpha$ leads to the *non*-unitary evolution

$$\frac{\partial \mathsf{A}_+}{\partial \alpha} \sim \frac{\partial \mathsf{W}}{\partial \alpha}\mathsf{A}_+ \,,$$

and *no* conserved probability density can be retrieved.

6.1.3 Barotropic Fluid Matter

In this section, we show how to add specific terms to the framework introduced previously in order to describe the presence of matter first in a non-supersymmetric manner and then in a *tentative* supersymmetric manner, e.g., the cosmological constant case (see Sect. 6.1.4 for the inclusion of a complex scalar field).

Non Supersymmetric Matter

Let us consider here a perfect fluid with barotropic equation of state as our matter field [29–31]. More precisely, the energy–momentum tensor is

$$T_{\mu\nu} = P g_{\mu\nu} + (P + \rho)\, u_\mu u_\nu, \tag{6.73}$$

where P, ρ, and u_μ are the pressure, energy density, and 4-velocity of the system, respectively. From

$$T^{\mu\nu}{}_{;\nu} = 0\ , \tag{6.74}$$

using a barotropic state function between the pressure and the energy density, i.e., $P = \gamma\rho$, with γ a constant, it follows that the energy density is given as a function of the scale factor $a = \mathrm{e}^{\alpha}$ of the FRW universe by

$$\rho = \frac{M_\gamma}{a^{3(\gamma+1)}}\ , \tag{6.75}$$

where M_γ is an integration constant. For a comoving fluid ($u_k = 0$), and the gauge $\mathcal{N}^k = 0$, this corresponds to the Lagrangian

$$L_{\text{matter}} \simeq \mathcal{N} M_\gamma a^{-3\gamma}\ . \tag{6.76}$$

Then the corresponding FRW Hamiltonian ($k = 0, \pm 1$) can be written

$$\mathcal{H} \simeq \frac{\pi_a^2}{24a} + 6ka - M_\gamma a^{-3\gamma}\ . \tag{6.77}$$

With

$$\pi_a \equiv -\mathrm{i}\hbar \frac{\partial}{\partial a}\ , \tag{6.78}$$

where $\Psi(a)$ is the wave function of the FRW universe model, the Wheeler–DeWitt equation is

$$\mathcal{H} \simeq \frac{1}{24a}\left(-\frac{\mathrm{d}^2}{\mathrm{d}a^2} + 144ka^2 - 384 M_\gamma a^{-3\gamma+1}\right)\ . \tag{6.79}$$

Physical states can be retrieved from the quantum constraints [29–31]

$$\overline{\mathcal{S}}\Psi = 0\,, \tag{6.80}$$

$$\mathcal{S}\Psi = 0\,, \tag{6.81}$$

and the wave function simply decomposes in the Grassmann variable representation as

$$\Psi = \mathsf{A}_+ + \mathsf{A}_-\theta^0\,, \tag{6.82}$$

where the component A_+ is the contribution of the bosonic sector, while A_- is the contribution of the fermionic sector.

Isospectral Factorization

In the following, we briefly review a method that is widely related to supersymmetric quantum mechanics (SQM) (see Sect. 3.3 of Vol. II). This is the *isospectral scheme* based on the *Riccati equation* [32–39]. There have been a few publications on this approach and it has recently been extended to SQC. More precisely, it usually requires nodeless, normalizable states of a Schrödinger equation. Moreover, SQM becomes in this context an equivalent formulation of the Darboux transformation method, wherein a transformation operator in the form of a differential operator intertwines two Hamiltonians and relates their eigenfunctions, generating a broader family of exactly solvable *new* potentials, starting from a given solvable local potential. This technique was applied to quantum cosmology in the context of the FRW cosmological models *without* matter field but in a radiation-filled FRW quantum universe, then extended to the barotropic FRW minisuperspace model, including a cosmological term [34–38].

The point here is that this technique basically generates, from known minisuperspace potentials (e.g., curvature and cosmological constant terms), *new* potentials that *lack an immediate physical significance*. What physical configuration or situation do the new potentials describe? It seems that this is completely arbitrary and may lack any physical correspondence with the observable universe. Nevertheless, it may prove useful in other contexts, although no fermions are explicitly involved. So let us review it here briefly.

The Hamiltonian is that of the barotropic FRW cosmological model, with cosmological constant

$$\mathcal{H}\Psi \simeq \frac{1}{24a}\left(-\frac{\mathrm{d}^2}{\mathrm{d}a^2} + 144ka^2 + 48\Lambda a^4 - 384M_\gamma a^{-3\gamma+1}\right)\Psi = 0\,, \tag{6.83}$$

and we choose to adopt [39] the semi-general (Hartle–Hawking) factor ordering, with

$$a^{-1}\frac{\mathrm{d}^2\Psi}{\mathrm{d}a^2} \longrightarrow a^{-1+\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}a}a^{-\mathrm{q}}\frac{\mathrm{d}\Psi}{\mathrm{d}a} = a^{-1}\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}a^2} - \mathrm{q}a^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}a}\right)\,, \tag{6.84}$$

where the (real) parameter q measures the ambiguity in the factor ordering. Then, the Wheeler–DeWitt equation becomes

$$\mathcal{H}_0\Psi = -a\frac{\mathrm{d}^2\Psi}{\mathrm{d}a^2} + \mathrm{q}\frac{\mathrm{d}\Psi}{\mathrm{d}a} + U(a)\Psi = 0\ , \tag{6.85}$$

where

$$U(a) = 144ka^3 + 48\Lambda a^5 - 384\pi G M_\gamma a^{-3\gamma+2}\ . \tag{6.86}$$

But in this approach we consider instead the (modified) equation

$$\mathcal{H}^+\Psi \equiv -a^{-2\mathrm{q}}\frac{\mathrm{d}^2\Psi}{\mathrm{d}a^2} + \mathrm{q}a^{-1-2\mathrm{q}}\frac{\mathrm{d}\Psi}{\mathrm{d}a} + a^{-1-2\mathrm{q}}U(a)\Psi = 0\ . \tag{6.87}$$

If follows that the first order differential operators

$$\mathcal{S}^+ \equiv -a^{-\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}a} + \mathsf{W}(a)\ , \tag{6.88}$$

$$\mathcal{S}^- \equiv a^{-\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}a} + \mathsf{W}(a)\ , \tag{6.89}$$

where W is the superpotential function, factorize the Hamiltonian (6.87) to give

$$\mathcal{H}^+ = \mathcal{S}^+\mathcal{S}^-\ . \tag{6.90}$$

The potential term $U(a)$ is related to the superpotential function $\mathsf{W}(a)_\gamma$ via the *Ricatti equation*,

$$U_+(a,\gamma) \equiv a^{1+2\mathrm{q}}W_\gamma^2 - a^{1+\mathrm{q}}\frac{\mathrm{d}W_\gamma}{\mathrm{d}a}\ , \tag{6.91}$$

with $U_+ \equiv U$.

Note 6.6 For $\mathrm{q} = -1$, (6.91) has close similarities with the relation determining the superpotential in Sect. 6.1.1.

In this supersymmetric factorization scheme, U_- is the partner superpotential of U_+ given by

$$\mathcal{H}^- \equiv \mathcal{S}^-\mathcal{S}^+\ , \qquad \mathcal{H}^- f(a) = 0\ , \tag{6.92}$$

where f is the wave function related to the Hamiltonian $\mathcal{H}^-$. Then the *isospectral potential* associated with $U_+(a,\gamma)$ is

$$U_-(a,\gamma) = a^{1+2\mathrm{q}}W^2 + a^{1+\mathrm{q}}W_\gamma' = U_+(a,\gamma) + 2a^{1+\mathrm{q}}W_\gamma'\ , \tag{6.93}$$

where $f(a)$ becomes

$$f(a) = W_\gamma u_\gamma + a^{-\mathrm{q}} u'_\gamma \ . \tag{6.94}$$

The point to emphasize is that, knowing the superpotential function, we can find the wave functions of the partner Hamiltonian $\mathcal{H}^-$.

In addition, and this is part of the usefulness of the method, the Ricatti equation admits a *more general* solution, viz.,

$$\hat{W} \equiv W_\gamma + \frac{1}{y_\gamma} \ , \tag{6.95}$$

which leads to a *Bernoulli equation* for y_γ, viz.,

$$y'_\gamma - 2W_\gamma a^{\mathrm{q}} y_\gamma = a^{\mathrm{q}} \ . \tag{6.96}$$

This in turn has solution

$$y_\gamma(a) = u_\gamma^{-2}\left(I_\gamma + \varpi\right) \ , \tag{6.97}$$

where $I_\gamma(a) = \int_0^A x^{\mathrm{q}} u_\gamma^2 \mathrm{d}x$. Hence,

$$\hat{W}(a) = W_\gamma + \frac{u_\gamma^2}{I_\gamma + \varpi} \ , \tag{6.98}$$

and the *entire* family of *new* bosonic potentials can therefore be retrieved as

$$\hat{U}_+(a,\gamma,\varpi) \equiv a^{1+2\mathrm{q}}\hat{W}^2(a,\gamma,\varpi) - a^{1+\mathrm{q}}\hat{W}'(a,\gamma,\varpi) \ , \tag{6.99}$$

with

$$\hat{U}_+(a,\gamma,\varpi) \equiv U_- - 2a^{-\mathrm{q}}\hat{W}' \tag{6.100}$$

$$= U_+(a,\gamma) - 4\frac{a^{1+\mathrm{q}} u_\gamma u'_\gamma}{I_\gamma + \varpi} + 2\frac{a^{1+2\mathrm{q}} u_\gamma^4}{(I_\gamma + \varpi)^2} \ . \tag{6.101}$$

Hence,

$$\hat{u}_\gamma \equiv g(\varpi)\frac{u_\gamma}{I_\gamma + \varpi} \tag{6.102}$$

is the isospectral solution for the new family potential (6.101), with the condition on the function $g(\varpi) = \sqrt{\varpi(\varpi+1)}$, and

$$\varpi \longrightarrow \pm\infty \ , \qquad g(\varpi) \longrightarrow \varpi \ , \qquad \hat{u}_\gamma \longrightarrow u_\gamma \ . \tag{6.103}$$

This ϖ parameter seems to point to some quantum cosmological dissipation (or damping).

Supersymmetry and the Cosmological Constant

So far, matter has been presented through a superpotential W depending on a and without any SUSY (fermionic) terms. In this section, a cosmological constant Λ term will be included in a supersymmetric manner, as indicated in [1, 16]. In fact, additional contributions are found in the form (thus going beyond a mere change in the superpotential $\mathsf{W}(a)$)

$$\mathcal{S}_\Lambda = \mathrm{i}\sqrt{2|\Lambda|}\psi^\Lambda , \tag{6.104}$$

$$\overline{\mathcal{S}}_\Lambda = \mp\mathrm{i}\sqrt{2|\Lambda|}\overline{\psi}_\Lambda , \tag{6.105}$$

where the upper (lower) sign applies to positive (negative) Λ, and where ψ^Λ and its adjoint $\overline{\psi}_\Lambda$ anticommute with the other variable ψ^X, $\overline{\psi}_Y$ and satisfy

$$\begin{aligned} &(\psi^\Lambda)^2 = 0 = (\overline{\psi}_\Lambda)^2 , \\ &\{\psi^\Lambda, \overline{\psi}_\Lambda\} = 1 , \end{aligned} \tag{6.106}$$

with the transformation

$$\delta\psi_\Lambda = -\mathrm{i}\sqrt{2\Lambda}\varepsilon , \qquad \delta\overline{\psi}_\Lambda = \mathrm{i}\sqrt{2\Lambda}\overline{\varepsilon} . \tag{6.107}$$

In this manner the Hamiltonian *properly* acquires the cosmological constant term $\mathcal{H}_\Lambda = \Lambda$.

The conserved fermion number $\mathcal{F}$ is now given by

$$\mathcal{F} \equiv \overline{\psi}_X\psi^X + \overline{\psi}_\Lambda\psi^\Lambda , \tag{6.108}$$

and a state in the *full* fermion sector (with $\mathcal{F} = f$) becomes

$$|f\rangle \equiv |\psi_f\rangle = \frac{1}{f!} f^{(0)}_{\mu_1\ldots\mu_f}(q)\overline{\psi}^{\mu_1}\ldots\overline{\psi}^{\mu_f}|0\rangle + \frac{1}{(f-1)!} f^{(1)}_{\mu_1\ldots\mu_{f-1}}(q)\overline{\psi}_\Lambda\overline{\psi}^{\mu_1}\ldots\overline{\psi}^{\mu_{f-1}}|0\rangle , \tag{6.109}$$

where we have transcribed into the notation of [1]. The constraint $\mathcal{S}|\psi_f\rangle = 0$ is now expressed by the equations (see (6.48) and (6.49))

$$\begin{aligned} &\left(\hbar\nabla_X - \frac{\partial W}{\partial q^X}\right) f^{(1)X}{}_{X_1\ldots X_{f-2}} = 0 , \\ &\left(\hbar\nabla_X - \frac{\partial \phi}{\partial q^X}\right) f^{(0)X}{}_{X_1\ldots X_{f-1}} + \sqrt{2|\Lambda|} f^{(1)}{}_{X_1\ldots X_{f-1}} = 0 , \end{aligned} \tag{6.110}$$

and $\overline{\mathcal{S}}|\psi_f\rangle = 0$ (generally) takes the explicit form

$$\varepsilon^{X_1\ldots X_{f+1}}\left(\hbar\frac{\partial}{\partial q^{X_1}} + \frac{\partial\phi}{\partial q^{X_1}}\right) f^{(1)}{}_{X_2\ldots X_f} \mp \sqrt{2|\Lambda|} f^{(0)}{}_{X_1\ldots X_f} = 0 , \tag{6.111}$$

where the doubled sign refers to the choice in (6.105).

For the case of a closed FRW universe, with $X \equiv \{a, \Lambda\}$, we have

$$\mathcal{S} = \mathrm{e}^{-3\alpha}\psi^0\left(-\mathrm{i}\frac{\partial}{\partial\alpha} + \mathrm{i}\mathrm{e}^{2\alpha}\right) + \mathrm{i}\sqrt{2|\Lambda|}\psi^\Lambda \,,$$
$$\overline{\mathcal{S}} = \mathrm{e}^{-3\alpha}\overline{\psi}^0\left(-\mathrm{i}\frac{\partial}{\partial\alpha} - \mathrm{i}\mathrm{e}^{2\alpha}\right) - \mathrm{i}\sqrt{2|\Lambda|}\overline{\psi}^\Lambda \,. \tag{6.112}$$

Writing the wave function as

$$\Psi = \mathsf{A}_+ + \mathsf{B}_0\theta^0 + \mathsf{B}_1^\Lambda\theta^\Lambda + \mathsf{A}_-\theta^0\theta^\Lambda \,, \tag{6.113}$$

with $\mathsf{A}_\pm$, B_0, B_1 as functionals of α, we get,[14] *in contrast to* the empty matter case (see Sect. 6.1.2),

$$\mathsf{A}_+ = \mathsf{A}_- = 0 \tag{6.114}$$

and

$$\left(-\frac{\partial}{\partial\alpha} + \mathrm{e}^{2\alpha}\right)\mathsf{B}_0 - \sqrt{2\Lambda}\mathrm{e}^{3\alpha}\mathsf{B}_1 = 0 \,,$$
$$\left(\frac{\partial}{\partial\alpha} + \mathrm{e}^{2\alpha}\right)\mathsf{B}_1 - \sqrt{2\Lambda}\mathrm{e}^{3\alpha}\mathsf{B}_0 = 0 \,. \tag{6.115}$$

Again bearing in mind the comments about having an imaginary or complex W in (6.42), a WKB type solution is possible:

$$\mathsf{B}_1 \simeq \left[\left(-2\Lambda\mathrm{e}^{2\alpha} + 1\right)^{-1/2} + 1\right]^{1/2}\exp\left[-\frac{1}{2}\int_0^{\exp(2\alpha)}\mathrm{d}u(-2\Lambda u + 1)^{1/2}\right] \,, \tag{6.116}$$
$$\mathsf{B}_0 \simeq \left[\left(-2\Lambda\mathrm{e}^{2\alpha} + 1\right)^{-1/2} - 1\right]^{1/2}\exp\left[-\frac{1}{2}\int_0^{\exp(2\alpha)}\mathrm{d}u(-2\Lambda u + 1)^{1/2}\right] \,, \tag{6.117}$$

This is *oscillatory* for $\mathrm{e}^{2\alpha} > (2\Lambda)^{-1}$ [16].

6.1.4 Complex Scalar Field

The content of this section is of particular relevance and should be contrasted with the content of Sect. 5.1.4, where we included supermatter (scalar fields) in an SQC

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[14] When $\Lambda = 0$, we recover the analysis in Exercise 6.2. In fact, for $\alpha \to -\infty$, near the initial singularity, the cosmological constant term is negligible and we get $\mathsf{B}_1 \to \mathrm{e}^{-\mathsf{W}}$, $\mathsf{W} = \mathrm{e}^{2\alpha}/2$ with $\mathsf{B}_0 \to 0$. The possibility of B_0 approaching e^{W} can be dealt with in a Bianchi IX setting (of which the FRW is a special case). If we require Ψ not to diverge for $\beta_\pm \to \infty$ for fixed α, then it must be excluded. Note that the remaining solution is *not* then the Hartle–Hawking state, but agrees with a Vilenkin type solution.

framework. Following [1, 4], we add a matter term $\mathcal{H}_{\rm M}$ to the Hamiltonian, in the form of a spatially homogeneous complex field $\phi(t)$, and also (on the basis of SUSY arguments) an arbitrary potential $\breve{V}(\phi)$ which is an analytic function of ϕ [the reader is invited to review the discussion of the Wess–Zumino model in Sect. 3.2.3, and consult (3.31), (3.32), (C.81), and Chap. 3 of Vol. II]:

$$\mathcal{H}_\phi = |\mathsf{p}_\phi|^2 + \Upsilon(\phi, \phi^*) \ , \tag{6.118}$$

$$\Upsilon = \mathrm{e}^{6\alpha+|\phi|^2}\Big[|\mathsf{D}\breve{V}(\phi)|^2 - 3|\breve{V}(\phi)|^2\Big] \ , \tag{6.119}$$

with $\mathsf{p}_\phi = \dot{\phi}^*$ as the canonically conjugate momentum of ϕ and

$$\mathsf{D}\breve{V}(\phi) \equiv \frac{\mathrm{d}\breve{V}}{\mathrm{d}\phi} + \phi^*\breve{V} \ .$$

A *useful choice* is the conformal gauge for the minisuperspace metric, which corresponds to cancelling the prefactor $\exp(6\alpha + |\phi|^2)$ in the potential term of (6.118), whence

$$\mathcal{H}_{\rm c} = \frac{1}{2}\mathcal{G}^{XY}(q)\left(\mathsf{p}_Y\mathsf{p}_X + \frac{\partial \mathsf{W}}{\partial q^Y}\frac{\partial \mathsf{W}}{\partial q^X}\right) + \mathrm{e}^{-6\alpha-|\phi|^2}|\mathsf{p}_\phi|^2 + \Big[|\mathsf{D}\breve{V}(\phi)|^2 - 3|\breve{V}(\phi)|^2\Big] \ , \tag{6.120}$$

with $\mathcal{G}_{XY}(q) = \mathrm{e}^{6\alpha+|\phi|^2}\mathcal{G}^{(0)}_{XY}$ (see (6.6)).

Concerning the supersymmetric procedure (characteristic of the approach in this chapter), besides the Grassmannian (odd) partners ψ^Y (and their adjoints) of the Bianchi variables q^X, we have two additional Grassmannian partners χ^1 and χ^2 (and their adjoints) for ϕ, $\overline{\phi}$, as well as another set χ^0, $\overline{\chi}^0$ (with the same properties as ψ^0, $\overline{\psi}^0$), all associated with the additional transformation

$$\phi \longrightarrow \phi + \mathrm{i}\chi^1\overline{\varepsilon}(t) \ . \tag{6.121}$$

The SUSY constraints $\mathcal{S}$ and $\overline{\mathcal{S}}$ are now extended by matter terms preserving the property $\mathcal{S}^2 = 0 = \overline{\mathcal{S}}^2$, so that again a supersymmetric quantization is obtained for the now matter-extended Hamiltonian. In particular, the supersymmetry constraints are written as

$$\mathcal{S} \equiv \mathcal{S}_0 + \mathcal{S}_{\rm K} + \mathcal{S}_{\rm P} \ , \tag{6.122}$$

$$\overline{\mathcal{S}} \equiv \overline{\mathcal{S}}_0 + \overline{\mathcal{S}}_{\rm K} + \overline{\mathcal{S}}_{\rm P} \ . \tag{6.123}$$

Here $\mathcal{S}_0$, $\overline{\mathcal{S}}_0$ are the pure gravitational terms (without cosmological constant) considered in Sect. 6.1.2. In addition, $\mathcal{S}_{\rm K}$, $\overline{\mathcal{S}}_{\rm K}$ in (6.123) are kinetic terms due to the scalar field, and $\mathcal{S}_{\rm P}$, $\overline{\mathcal{S}}_{\rm P}$ are associated with a potential-like expression.

Note 6.7 Before proceeding, we should comment on the choice of prefactor in $\mathcal{G}_{XY}$. In fact, it is just a matter of convenience. If we opt instead for $\mathcal{G}_{XY}(q)\left(= \mathcal{G}^{(0)}_{XY}\right) = \eta_{XY}$, the Hamiltonian $\mathcal{H}_{\rm c}$ and $\mathcal{S}_0$, $\overline{\mathcal{S}}_0$ are then simpler because *no* connection terms appear for a flat metric. Furthermore, the kinetic matter terms $\mathcal{S}_{\rm K}$, $\overline{\mathcal{S}}_{\rm K}$ have no 3-fermion terms, because the extension of minisuperspace by ϕ and ϕ^* leaves the extended metric flat. However, the potential terms $\mathcal{S}_{\rm P}$, $\overline{\mathcal{S}}_{\rm P}$ now become more complicated. The additional dependence of $\mathcal{H}_{\rm M}$ on α and $|\phi|^2$ implies more 3-fermion terms which do not have any obvious interpretation in terms of (minisuperspace) spin connections.

Note 6.8 The following may provide a clearer view:

- The kinetic terms involve considering ϕ as an *additional complex coordinate* in minisuperspace at the same level as, e.g., the real $\beta_\pm$, and extending it to a ϕ, α dependent metric $\mathcal{G}_{XY}$, an affine connection Γ^Y_{XZ}, the vielbein $e^x{}_X$ and spin connection ω_{Yxy} (treating ϕ^* as a constant parameter).
- $\mathcal{S}_{\rm K}$ contains a new fermionic field χ^1 associated with ϕ (and its adjoint $\overline{\chi}_1$, with fermionic commutation relations).
- The derivative operators $\partial/\partial\phi$ and spin connection terms follow from the extension of minisuperspace by ϕ, and the fact that, in our conformal gauge, $\mathcal{G}_{XY}$ depends also on ϕ (while ϕ^* merely plays the role of a parameter in $\mathcal{S}_{\rm K}$).
- Thus, we have

$$\mathcal{S}_{\rm K} = {\rm i}e^{-3\alpha-|\phi|^2/2}\chi^1\left(-\sqrt{2}\hbar\frac{\partial}{\partial\phi} - 3\hbar\overline{\chi}_1\psi^0 - \hbar\frac{\phi^*}{\sqrt{2}}\psi^x\overline{\psi}_x\right) ,$$
$$\overline{\mathcal{S}}_{\rm K} = {\rm i}e^{-3\alpha-|\phi|^2/2}\overline{\chi}_1\left(-\sqrt{2}\hbar\frac{\partial}{\partial\phi^*} - 3\hbar\overline{\psi}_0\chi^1 - \hbar\frac{\phi}{\sqrt{2}}\overline{\psi}_x\psi^x\right) . \quad (6.124)$$

- Moreover, $\mathcal{S}_0 + \mathcal{S}_{\rm K}$ and $\overline{\mathcal{S}}_0 + \overline{\mathcal{S}}_{\rm K}$ generate suitable supersymmetry transformations and algebra when the full Hamiltonian $\mathcal{H}$ reduces to $\mathcal{H}_{\rm c}$, with vanishing matter potential in the classical limit [1, 3, 4].
- Finally, $\mathcal{S}_{\rm P}$ and $\overline{\mathcal{S}}_{\rm P}$ in (6.123) are potential terms due to the scalar field. Morever, we can make the following points:
  - $\mathcal{S}_{\rm P}$ contains another fermionic field χ^2 associated with ϕ, and $\overline{\mathcal{S}}_{\rm P}$ contains its adjoint $\overline{\chi}_2$.
  - An additional fermionic variable χ^0 associated with $q^0 = \alpha$ also appears in $\mathcal{S}_{\rm P}$, together with its adjoint $\overline{\chi}_0$. It is made necessary by the negative term in the matter potential of (6.118).
  - A point to emphasize is that, unlike the 3-fermion terms in $\mathcal{S}_0$ and $\mathcal{S}_{\rm K}$, the ones in (6.126) have no *geometrical* interpretation in terms of a spin

connection in minisuperspace, but would have within a Kähler geometry for ϕ (see Sect. 3.3.1).

– Explicitly [1, 4],

$$\mathcal{S}_{\mathrm{P}} = \mathrm{i}\sqrt{2}\chi^2\left[\left(\mathsf{D}\check{V}(\phi)\right)^* + \frac{\hbar}{\sqrt{3}}\mathrm{e}^{-3\alpha-|\phi|^2/2}\overline{\chi}_0\chi^1\right] + \mathrm{i}\sqrt{6}\chi^0\left(\check{V}(\phi)\right)^* , \quad (6.125)$$

$$\overline{\mathcal{S}}_{\mathrm{P}} = \mathrm{i}\sqrt{2}\overline{\chi}_2\left[-\mathsf{D}\check{V}(\phi) + \frac{\hbar}{\sqrt{3}}\mathrm{e}^{-3\alpha-|\phi|^2/2}\overline{\chi}_1\chi^0\right] + \mathrm{i}\sqrt{6}\overline{\chi}_0\check{V}(\phi) . \quad (6.126)$$

Solutions from the Quantum Constraints

The conserved fermion number takes the form

$$\mathcal{F} = \overline{\psi}_x\psi^x + \overline{\chi}^0\chi_0 + \overline{\chi}_1\chi^1 + \overline{\chi}_2\chi^2 , \quad (6.127)$$

now with *seven* sectors for $\mathcal{F} = f$, with $f = 0, 1, 2, 3, 4, 5, 6$. State solutions can be found, e.g., in the framework indicated in Sect. 6.1.2. In particular, nontrivial solutions in the sectors $f = 0$ and $f = 6$ exist only if $\check{V}(\phi) \equiv 0$, and are then given by (see Sect. 6.1.3 for the presence of a cosmological constant and Sect. 3.3 of Vol. II for more notation)

$$|\Psi_0\rangle \equiv f(\phi)\mathrm{e}^{-\mathsf{W}/\hbar}|0\rangle , \quad (6.128)$$

$$|\Psi_6\rangle \equiv g(\phi^*)\mathrm{e}^{\mathsf{W}/\hbar}|6\rangle , \quad (6.129)$$

with arbitrary analytical functions $f(\phi)$, $g(\phi^*)$. Furthermore, note the following points:

- The first order fermion sector (see (6.109)) is found to be

$$|1\rangle \equiv |\Psi_1\rangle = \overline{\mathcal{S}} f(q,\phi,\phi^*)\mathrm{e}^{-\mathsf{W}/\hbar}|0\rangle .$$

  It satisfies $\overline{\mathcal{S}}|\Psi_1\rangle = 0$, and the remaining condition $\mathcal{S}|\Psi_1\rangle = 0$ leads to the Wheeler–DeWitt equation for $f(q,\phi,\phi^*)$, viz.,

$$\frac{\hbar}{2}\eta^{XY}\left(\hbar\frac{\partial}{\partial q^X} - 2\frac{\partial\mathsf{W}}{\partial q^X}\right)\frac{\partial f}{\partial q^Y} - 3\hbar^2\frac{\partial f}{\partial\alpha} + \hbar^2\frac{\partial^2 f}{\partial\phi\partial\phi^*} + \hbar^2\phi^*\frac{\partial f}{\partial\phi^*} = \mathrm{e}^{6\alpha+|\phi|^2}\left(|\mathsf{D}\check{V}|^2 - 3|\check{V}|^2\right)f . \quad (6.130)$$

- For $f = 2$, the ansatz

$$|2\rangle| \equiv \Psi_2\rangle = \overline{\mathcal{S}}\left(f^X\overline{\psi}_X + g_1\overline{\chi}_1 + g_2\overline{\chi}_2\right)|0\rangle \quad (6.131)$$

can be made, with five undetermined functions f^X, g_1, and g_2. Terms with $\overline{\mathcal{S}}\overline{\chi}_0$ have been eliminated from (6.131) by subtracting the vanishing state $\overline{\mathcal{S}}^2 h(q)|0\rangle = 0$ with appropriately chosen $h(q)$. We obtain a system of four Wheeler–DeWitt equations, i.e., second order wave equations, and two auxiliary equations which are of first order. One of the auxiliary equations is not independent, however, and may be dropped.

- Similarly for $f = 3$, the ansatz

$$|3\rangle \equiv |\Psi_3\rangle = \overline{\mathcal{S}}\left(f^{XY}\overline{\psi}_X\overline{\psi}_Y + g_1^X\overline{\psi}_X\overline{\chi}_1 + g_2^X\overline{\psi}_X\overline{\chi}_2 + g_{12}\overline{\chi}_1\overline{\chi}_2\right)|0\rangle \tag{6.132}$$

can be made with 10 undetermined functions $f^{XY} = -f^{YX}$, g_1^X, g_2^X, and g_{12}. Once again we have made use of $\overline{\mathcal{S}}^2 = 0$ to eliminate the five terms involving $\overline{\mathcal{S}}\overline{\chi}_0$. Now a system of 6 second order wave equations is obtained, together with 9 auxiliary equations of first order, five of which are not independent and may be dropped.

Let us now analyse the 1-fermion sector in more detail for the case of a Bianchi IX model. The potential W for $|\beta_\pm| \to \infty$ diverges. This can be avoided if f is independent of α and $\beta_\pm$ in that limit. So this becomes a boundary condition: for $\alpha \to -\infty$ at fixed $\beta_\pm$, ϕ, and ϕ^*, the potential W becomes less relevant. Then W vanishes in that limit. Hence, for α negative and sufficiently large, the right-hand side of (6.130) may be negligible, and W also approaches zero. The amplitude f satisfies a free wave equation. Either $f(q,\phi,\phi^*) \to f(\phi)$ or one of $\partial f/\partial q^Y$, $\partial f/\partial\phi$ differs from zero. In the former case,

$$|\psi_1\rangle \simeq \mathrm{e}^{-\mathsf{W}} f(\phi)\left[-\mathsf{D}\check{V}(\phi)\overline{\chi}^2 + \sqrt{3}\check{V}(\phi)\overline{\chi}_0\right]|0\rangle\ , \tag{6.133}$$

being arbitrarily anisotropic ($\mathsf{W} = 0$) at large negative α but becoming isotropic (located at $\beta_\pm \simeq 0$). In the latter case,

$$|\psi_1\rangle \simeq \mathrm{e}^{-\mathsf{W}-3\alpha-\phi^2/2}\left(\frac{\partial f}{\partial q^Y}\psi^Y + \sqrt{2}\frac{\partial f}{\partial\phi^*}\chi_1\right)|0\rangle\ , \tag{6.134}$$

again being arbitrarily anisotropic ($\mathsf{W} = 0$) at large negative α, but becoming isotropic (located at $\beta_\pm \simeq 0$).

For sufficiently large α, (6.130) may be solved in a Born–Oppenheimer approximation (see also Chap. 2 of Vol. II), *assuming* that the scalar field adjusts itself almost instantaneously to the gravitational field. One possible approach is to consider the case where $\check{V}(\phi)$ has a quadratic stationary point ϕ_0 with $\check{V}(\phi_0) = 0$. Hence,

$$|\Psi_1\rangle \simeq f(q,\phi,\phi^*)\mathrm{e}^{-\mathsf{W}}\left[-\mathsf{D}\check{V}(\phi)\overline{\chi}^2 + \sqrt{3}\check{V}(\phi)\overline{\chi}_0\right]|0\rangle\ , \tag{6.135}$$

with the function f becoming localized at ϕ values where $|\mathsf{D}\check{V}| = \sqrt{3}|\check{V}|$ (due to the growing prefactor), with $f\mathrm{e}^{-\mathsf{W}}$ oscillatory. To be more precise, the total wave function in this limit is approximately

$$f(\phi, \phi^*, q) \simeq \tilde{f}(\phi, \phi^*, \alpha) g(\alpha, \beta_+, \beta_-) \,. \tag{6.136}$$

In the limit where an adiabatic approximation is valid (see (6.29) and (6.30)), the prefactor $\mathrm{e}^{-\mathsf{W}/\hbar}$ or $\mathrm{e}^{-\tilde{\mathsf{W}}/\hbar}$ of f will then be very sharply peaked at $\beta_+ = 0 = \beta_-$. Therefore we may put $\beta_+ = \beta_- = 0$ in f, and in the semi-classical limit the solutions are given by

$$g \simeq (2E_0\mathrm{e}^{3\alpha} - \mathrm{e}^{4\alpha})^{1/2} \exp\left[\frac{1}{2}\mathrm{e}^{2\alpha} - \mathrm{i}\int^{\alpha} \mathrm{d}\alpha (2E_0\mathrm{e}^{3\alpha} - \mathrm{e}^{4\alpha})^{1/2}\right] \,, \tag{6.137}$$

for $\phi = \phi_{IX}$, or

$$g \simeq (2E_0\mathrm{e}^{3\alpha} - \mathrm{e}^{4\alpha})^{1/2} \exp\left[-\frac{1}{2}\mathrm{e}^{2\alpha} - \mathrm{i}\int^{\alpha} \mathrm{d}\alpha (2E_0\mathrm{e}^{3\alpha} - \mathrm{e}^{4\alpha})^{1/2}\right] \,, \tag{6.138}$$

for $W = \tilde{\mathsf{W}}_{IX}$, which are both outgoing waves as long as

$$\mathrm{e}^{\alpha} < 2E_0 \,. \tag{6.139}$$

In the semi-classical limit, it follows that

$$f\mathrm{e}^{-\mathsf{W}} \sim \mathrm{e}^{\mathrm{i}S(q,\phi,\phi^*)/\hbar} \,. \tag{6.140}$$

Thus for $\alpha \to \infty$, we obtain a wave packet moving along trajectories of the classical system. The classical trajectories will start with 'initial' conditions in that region of configuration space where the wave function makes its transition from an exponentially decaying or growing behavior $\sim \mathrm{e}^{-\mathsf{W}/\hbar}$ or $\mathrm{e}^{-\tilde{\mathsf{W}}/\hbar}$ to an oscillatory behavior $\mathrm{e}^{\mathrm{i}S/\hbar}$. However, the attentive explorer may have noted the following point. In fact, the probability density associated with a (conserved) Klein–Gordon type density associated with a Wheeler–DeWitt equation is zero unless tunneling becomes possible, due to a cosmological constant or coupling to a massive scalar field (as extracted here), thus rendering the solution Ψ with complex components. The point, however, is that the quantized $\mathcal{H}$ differs from the classical $\mathcal{H}_c$ by a 'spin' term and this *should* disappear in the classical limit as $\hbar \to 0$.

6.1.5 Gauge Fields

In this section, we apply the methods of 'hidden' $N = 2$ SQC (but with some modifications) to coupled SU(2) Einstein–Yang–Mills (EYM) systems in axially symmetric Bianchi cosmological models [2]. The reader may consider comparing some of what follows with Sect. 5.1.5. The interest in classical EYM systems is that the nonlinear nature of the source (the YM field) produces nontrivial spacetime configurations mainly in strong field regions, i.e., near cosmological spacetime singularities, where a purely classical description of spacetime should be replaced by a quantum field theory.

The construction of the quantum Hamiltonian here is Hermitian self-adjoint for any type of signature of the metric in minisuperspace. Quantum states are those of null fermion and filled fermion sectors, while in other fermion sectors they exist only if the manifold, determined by the minisuperspace metric, has corresponding nontrivial *cohomologies* (see Sect. 3.3 of Vol. II, which is relevant, although not crucial, for what follows).

For homogeneous axially symmetric SU(2) EYM systems, we take the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[R - \frac{1}{2}f^{(a)}_{\mu\nu}f^{(a)\mu\nu}\right] , \tag{6.141}$$

and restrict it to homogeneous spacetimes of an axially symmetric Bianchi type. The general diagonal Bianchi type axially symmetric spacetimes are parametrized by two independent functions of a cosmological time $b_1(t) = b_2(t)$ and $b_3(t)$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\Big[(\omega^1)^2 + (\omega^2)^2\Big] + b_3^2(t)(\omega^3)^2 , \tag{6.142}$$

where the ω^i constitute a basis of left-invariant one-forms for the spatially homogeneous three-metrics.

The general ansatz for an SU(2) YM field (see also Sect. 5.1.5 and [40, 41]), compatible with the symmetries of axially symmetric Bianchi-type cosmological models, can be displayed in terms of two independent real-valued functions $\alpha(t)$ and $\gamma(t)$:

$$\alpha(t)\big(\omega^1\mathcal{T}_1 + \omega^2\mathcal{T}_2\big) + \gamma(t)\omega^3\mathcal{T}_3 , \tag{6.143}$$

where $\mathcal{T}_i$ are SU(2) group generators, normalized so that $[\mathcal{T}_i, \mathcal{T}_j] = \varepsilon_{ijk}\mathcal{T}_k$. In particular, $\mathcal{G}_{XY}(q)$ is taken as the metric of the *extended* minisuperspace of spatially homogeneous axially symmetric three-metrics coupled with the corresponding SU(2) Yang–Mills fields, i.e., $q^X = (b_1, b_3, \alpha, \gamma)$ are bosonic components of a *superfield*. Then, the same number of fermionic fields ($\bar{\xi}^X$ and ξ^X) is introduced, thereby implementing an $N = 2$ supersymmetrization, provided that the potential $U(q)$ satisfies

$$U(q) \equiv \frac{1}{2}\mathcal{G}^{XY}(q)\frac{\partial \mathsf{W}(q)}{\partial q^X}\frac{\partial \mathsf{W}(q)}{\partial q^Y} . \tag{6.144}$$

Note 6.9 We note that the resulting superpotentials W are direct sums of purely gravitational W_{gr} and Yang–Mills parts W_{YM}. Since the superpotential is the *least* Euclidean action (and solution of the Euclidean Hamilton–Jacobi equation), the corresponding Euclidean solutions will give the main contribution to the wave function in a quasiclassical approach. Therefore, the gravitational part of the superpotential W_{gr} determines the Euclidean gravitational background configurations, which will not change *if* matter configurations do *not* contribute to the energy–momentum tensor. The Yang–Mills part W_{YM} of the superpotential fulfills this by leading to self-dual YM instantons, where the energy–momentum tensor vanishes identically. In addition, the Yang–Mills part of the superpotential coincides with the corresponding Chern–Simons functional (see Chap. 6 of Vol. II).

From the supersymmetry transformations (see also Sect. 3.3 of Vol. II)

$$\begin{aligned} \delta q^X &= \bar{\varepsilon}\xi^X - \varepsilon\bar{\xi}^X , \\ \delta\xi^X &= \varepsilon(-\mathrm{i}\dot{q}^X + \Gamma^X_{YZ}\bar{\xi}^Y\xi^Z - \partial^X\mathsf{W}) , \\ \delta\bar{\xi}^X &= \bar{\varepsilon}(\mathrm{i}\dot{q}^X + \Gamma^X_{YZ}\bar{\xi}^Y\xi^Z - \partial^X\mathsf{W}) , \end{aligned} \tag{6.145}$$

and in order to prevent $N = 2$ SUSY breaking at the classical level, the classical pure bosonic configurations must satisfy

$$\dot{q}^X(t) = 0 , \qquad \partial^X\mathsf{W}\Big[q^Y(t)\Big] = 0 , \tag{6.146}$$

along with the classical Hamiltonian constraint $\mathcal{H}_{\text{c}} = 0$, e.g., for a flat spacetime, a scalar rest particle ($\dot{q}^X = 0$) at the bottom of a potential with $U(q^X) = 0$.

In addition, it is worth noting the following points:

- For homogeneous systems with gravity included, any nontrivial classical solution of Einstein's equations (or Einstein's equations coupled with matter) is *never* such that all the momenta vanish, i.e., $\dot{q}^X(t) \neq 0$. These systems satisfy $\mathcal{H} = 0$ due to the dynamical balance between the kinetic and potential terms with both positive and negative signs. Hence, any homogeneous Einstein (or Einstein–matter) system embedded in the $N = 2$ supersymmetric (sigma) model *never* has solutions for the equations of motion with unbroken supersymmetry, i.e., supersymmetry is always spontaneously broken at the tree level.
- However, in the quantum mechanical perspective it may be different:
  - Let us consider solutions of the zero energy Schrödinger-type equation $\mathcal{H} = 0$ in the empty and filled fermion sectors.
  - The superpotential $\mathsf{W}(q)$ corresponds to the solutions in empty, $\mathrm{e}^{-\mathsf{W}}$, and filled, e^{W}, fermion sectors, when we find the supersymmetric wave functions.

We further adopt the definition in [2] for the norm of the physical state $\langle f \rangle$ as $\pm \int \sqrt{|-g|}\langle f||f\rangle \mathrm{d}^4 q$ in order to avoid the problem of the negative norm in the four-fermion sector, caused by the timelike component of the fermionic field. The plus sign in the definition of the norm corresponds to $+\mathsf{W}(q)$, while the minus sign has to be taken as $-\mathsf{W}(q)$.

For the positive sign of the superpotential and pure gravitational systems, when the α and γ functions along with their fermionic partners are set equal to zero, quantum mechanically, the supersymmetry is restored for Bianchi type I, II, IX, Kantowski–Sachs, and FRW models, since the solution of $\mathcal{H} = 0$, $|f_0^{\mathrm{gr}}\rangle \simeq \mathrm{e}^{-\mathsf{W}_{\mathrm{gr}}}|0\rangle$, in the null fermion sector is normalizable:

$$\int_0^{+\infty} \mathrm{d}b_1 \int_0^{+\infty} \mathrm{d}b_3 \sqrt{|-g|}\mathrm{e}^{-2\mathsf{W}_{\mathrm{gr}}} < \infty\,. \tag{6.147}$$

Contrary to standard supersymmetric quantum mechanics (SQM), in the EYM setting, the supersymmetry is spontaneously broken at the tree level and is then restored quantum mechanically. The only exceptions are a second superpotential for Bianchi type IX and Bianchi type VIII, where the supersymmetry remains broken at the quantum level as well, since their norm (6.147) diverges at the upper limit.

Furthermore, inclusion of the Yang–Mills field spontaneously breaks the supersymmetry again, because the Yang–Mills part of the superpotential W_{YM} for all considered models (being the corresponding Chern–Simons term), is an odd function of α and γ. Consequently, the YM parts of the wave function $|f_0^{\mathrm{YM}}\rangle \simeq \mathrm{e}^{\pm\mathsf{W}_{\mathrm{YM}}}|0\rangle$, both in null and filled fermion sectors, are not normalizable:

$$\int_{-\infty}^{+\infty} \mathrm{d}\alpha \int_{-\infty}^{+\infty} \mathrm{d}\gamma \sqrt{|-g|}\mathrm{e}^{\pm 2\mathsf{W}_{\mathrm{YM}}} \longrightarrow \infty\,. \tag{6.148}$$

Hence, after quantization, the only nontrivial zero-energy wave functions in null and filled fermion sectors turn out to have a diverging norm, and this fact indicates spontaneous breaking of supersymmetry (see Sect. 3.3 of Vol. II), caused by YM instantons. The spontaneous supersymmetry breaking which takes place if the Yang–Mills field is added to pure gravity is caused (in a quasiclassical description) by a YM instanton contribution to the wave function.

6.2 Superstring Scenarios and Duality Transformations

This will be another long section, but whose content will be quite significant, for the following reasons. First, it brings the framework of superstring theory [42–47] into the *direct* application of SQC methodology. So far, we have been using either general relativity to obtain an $N = 2$ SUSY minisuperspace (see the last section) or 4D $N = 1$ SUGRA. Second, we will show how specific transformation properties known as *dualities*, which leave the string action invariant, can be useful when implementing SUSY minisuperspaces.

In addition, it should be noted that this symmetry forms the basis of the pre-big bang inflationary scenario [48] and that its origin can be traced to the T-duality of string theory. The consequences of scale factor duality for string quantum cosmology have been explored by a number of authors [5, 6]. In particular, it has been pointed that the duality is related to a hidden supersymmetry [7]. This is an important feature to explore within our quantum cosmological models.

Finally, we will find plenty of uncharted domains to venture into. Indeed, the vast majority of published works [49–53, 5–8] has dealt only with implementing $N = 2$ SUSY, using the methodology of Sect. 6.2. However, a full dimensional reduction from a 10D superstring theory to an $N = 4$ minisuperspace would be the best route, although it has not yet been worked out.

To begin with, we summarize the derivation of the effective action presented by Maharana and Schwarz [54] for the *heterotic* string case, and then derive the Wheeler–DeWitt and corresponding SUSY constraint equations.

6.2.1 N=2 SUSY Bianchi Models from the Heterotic String

In the vast published literature on string (quantum) cosmology [55, 48, 49, 53, 5–7, 56] *only* the bosonic sector has been employed, neglecting the contribution from the fermionic terms to the dynamics of the very early (quantum) universe.

In order to get a taste of what fermion sectors may introduce into the physical (cosmological) description, let us recall that all 10D superstring theories contain in their bosonic sector a dilaton, a graviton, and an antisymmetric 2-form potential in the Neveu–Schwarz/Neveu–Schwarz (NS–NS) sector of the theory. Moreover, most of the studied cosmologies have $\hat{d}$ compact Abelian isometries, whence the NS–NS sector of the effective action has been compactified on a $\hat{d}$-torus. The reduced action is then *invariant* under a global $\mathrm{O}(\hat{d},\hat{d})$ T-duality, where the scalar fields parametrize the coset $\mathrm{O}(\hat{d},\hat{d})/[\mathrm{O}(\hat{d}) \times \mathrm{O}(\hat{d})]$. This leads to an $\mathrm{O}(\hat{d},\hat{d})$ invariant Wheeler–DeWitt equation. In particular, restricted to spatially flat, isotropic FRW cosmologies, with the dilaton–graviton sector of the string effective action invariant under an inversion of the scale factor and a shift in the dilaton field [48, 56]. This *scale factor duality* is a subgroup of T-duality and can lead to a supersymmetric extension of quantum cosmology, where the classical minisuperspace Hamiltonian corresponds (at the quantum level) to the bosonic component of an $N = 2$ supersymmetric Hamiltonian [5–7].

But let us proceed with a more detailed description. At the tree level, the action of the bosonic sector of the $\check{D}$-dimensional heterotic string is

$$S_{\check{g}} = \int_{\mathcal{M}} \mathrm{d}^{\check{D}} x \sqrt{\check{g}} \mathrm{e}^{-\check{\phi}} \left[\check{R}(\check{g}) - (\check{\nabla}\check{\phi})^2 + \frac{1}{12} \check{H}_{\check{\mu}\check{\nu}\check{\rho}} \check{H}^{\check{\mu}\check{\nu}\check{\rho}} \right] , \tag{6.149}$$

where the check henceforth denotes quantities in a $\check{D}$-dimensional spacetime $\mathcal{M}$, with $\check{D} \equiv 1 + D + \hat{d}$, and $D \equiv 3$. Then $\check{\phi}$ is the dilaton (scalar) field and $\check{H}$ is the totally antisymmetric 3-index field. The universe is considered as the product space

$\mathcal{M} = \mathcal{J} \times \mathcal{K}$, where the $(D+1)$-dimensional spacetime $\mathcal{J}(x^\rho)$ has metric $g_{\mu\nu}(x^\rho)$ and the $\hat{d}$-dimensional internal space $\mathcal{K}(y^{\hat{\alpha}})$ must be Ricci flat *if* the matter fields are independent of its coordinates $y^{\hat{\alpha}}$. This corresponds to the Calabi–Yau spaces, often considered in string theory.

In simple descriptions, however, it is sufficient to assume that $\mathcal{K}$ is a torus, viz., $\mathcal{K} = S^1 \times S^1 \times \ldots \times S^1$. The complete metric on $\mathcal{M}$ is then given by

$$\breve{g}_{\breve{\mu}\breve{\nu}} = \begin{pmatrix} g_{\mu\nu} + A_\mu{}^\gamma A_{\nu\gamma} & A_{\mu\hat{\beta}} \\ A_{\nu\hat{\alpha}} & \hat{\mathsf{g}}_{\hat{\alpha}\hat{\beta}} \end{pmatrix} , \tag{6.150}$$

where $\hat{\mathsf{g}}_{\hat{\alpha}\hat{\beta}}$ is the metric[15] on $\mathcal{K}$. Subsequently, the dimensionally reduced effective action in $(D+1)$-dimensions becomes (see [48] for more details)

$$S = \int \mathrm{d}^{D+1}x\sqrt{g}\mathrm{e}^{-\phi}\Big[-R - (\nabla\phi)^2 + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} \\ - \frac{1}{8}\mathrm{Tr}\left(\nabla_\mu \mathsf{M}^{-1}\nabla^\mu \mathsf{M}\right) + \frac{1}{4}F_{\hat{a}\mu\nu}F^{\mu\nu\hat{a}}\Big] , \tag{6.151}$$

where

$$H_{\mu\nu\rho} \equiv \nabla_\mu B_{\nu\rho} - \frac{1}{2}A_{\mu\hat{a}}F^{\hat{a}}_{\nu\rho} + (\text{cyclic}) , \tag{6.152}$$

$$F^{\hat{a}}_{\mu\nu} \equiv \nabla_\mu A^{\hat{a}}_\nu - \nabla_\nu A^{\hat{a}}_\mu , \tag{6.153}$$

and

$$\phi = \breve{\phi} - \frac{1}{2}\ln\det \mathrm{g} \tag{6.154}$$

is the shifted dilaton field, with B denoting a (NS) 2-form field. We henceforth drop the hat on $\hat{d}$-dimensional quantities. The $2d \times 2d$ matrix M is defined by

$$\mathsf{M} \equiv \begin{pmatrix} \mathrm{g}^{-1} & -\mathrm{g}^{-1}B \\ B\mathrm{g}^{-1} & \mathrm{g} - B\mathrm{g}^{-1}B \end{pmatrix} , \tag{6.155}$$

with

$$\breve{\eta} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \mathsf{M}^{-1} \equiv \breve{\eta}\mathsf{M}\breve{\eta} , \tag{6.156}$$

from which it follows that

[15] Also representing the $\breve{d}(\breve{d}+1)/2$ modulus (degrees of freedom) fields.

$$\mathsf{M}^{\mathrm{T}}\breve{\eta}\mathsf{M} = \breve{\eta} \;, \tag{6.157}$$

implying that $\mathsf{M} \in \mathrm{O}(d, d)$. Hence, since $g_{\mu\nu}$ and ϕ are also invariant under the action of this group, the dimensionally reduced action is symmetric under a *global* $\mathrm{O}(d, d)$ transformation.

Following [48, 56], we can further impose $H_{\mu\nu\rho} = 0$ and $A_\mu^{\hat{a}} = 0$. If all the fields are constant on the spatial hypersurfaces of $\mathcal{J}$, the $d \times d$ matrix $\mathrm{g} + B$ becomes

$$\mathrm{g} + B = \mathrm{diag}\left(\mathcal{E}_1, \ldots, \mathcal{E}_{d/2}\right) \;, \qquad \mathcal{E}_j = \begin{pmatrix} \mathrm{e}^{\varphi_j} & \varsigma_j \\ -\varsigma_j & \mathrm{e}^{\varphi_j} \end{pmatrix} \;, \tag{6.158}$$

where g and B are spacetime dependent and we assume that d is even. The action (6.151) then simplifies to

$$S = \int \mathrm{d}^{D+1}x\sqrt{g}\mathrm{e}^{-\phi}\left\{R - (\nabla\phi)^2 + \frac{1}{2}\sum_{j=1}^{d/2}\left[(\nabla\varphi_j)^2 + \mathrm{e}^{-2\varphi_j}(\nabla\varsigma_j)^2\right]\right\} \;. \tag{6.159}$$

We now focus our attention on spatially homogeneous models (see Sects. 6.1.1 and 6.1.2). The essential element is the following. After the compactification of (6.151) onto a 6-torus, the effective 4D action (6.159) resembles a scalar–tensor theory. In this context, the Wheeler–DeWitt equation can be retrieved:

- Directly by calculating the Hamiltonian in terms of the conjugate momenta and imposing the canonical commutation relations in the usual manner.

Or *equivalently*:

- The classical action is first transformed into the Einstein–Hilbert form *before* quantization, through a conformal transformation. This means using a *different* set of variables to be quantized and consequently a *different* Wheeler–DeWitt equation. This approach involves the following:
  - Apply Einstein gravity *directly*.
  - The Wheeler–DeWitt equation takes a much simpler form.
  - After the *conformal* Wheeler–DeWitt equation has been solved, the results should be transformed back into the original coordinates to discuss the solution in the original frame.

Taking $D = 3$, $d = 6$ as our case study [49–52], and a conformal transformation

$$\tilde{\Omega}^2 \equiv \mathrm{e}^{-\phi} \;, \tag{6.160}$$

the transformed action is

$$S = \int \mathrm{d}^4x\sqrt{\tilde{g}}\left\{\tilde{R} - \frac{1}{2}\left(\tilde{\nabla}\phi\right)^2 + \frac{1}{2}\sum_{j=1}^{3}\left[\left(\tilde{\nabla}\varphi_j\right)^2 + \mathrm{e}^{-2\varphi_j}\left(\tilde{\nabla}\varsigma_j\right)^2\right]\right\} \;, \tag{6.161}$$

where $\tilde{g}$ is the conformally transformed metric and the rescaled dilaton is constant on the surfaces of homogeneity. The transformed line element is

$$d\tilde{s}^2 = -d\eta^2 + \tilde{h}_{ij}\omega^i\omega^j \ , \tag{6.162}$$

where

$$\tilde{h}_{ij} = e^{2\tilde{\alpha}}\left(e^{2\beta}\right)_{ij} = e^{-\phi+2\alpha}\left(e^{2\beta}\right)_{ij} \tag{6.163}$$

is the *re*scaled 3-metric, $\eta \equiv \int dt\tilde{\Omega}(t)$ is the time in the conformal frame, and

$$\tilde{\alpha} \equiv \alpha + \ln\tilde{\Omega} \ . \tag{6.164}$$

Empty and Filled Fermionic Sectors (Bianchi Class A)

Let us describe the above minisuperspace in a more compact manner, spanned by 10 coordinates $q^X = (\tilde{\alpha}, \beta_\pm, \phi, \varphi_j, \varsigma_j)$, where $(X = 0, 1, 2, \ldots 9)$. (Here we identify q^0 with $\tilde{\alpha}$, q^1 with β_+, etc.) Hence the classical Hamiltonian has the form

$$\mathcal{H} \propto -\mathsf{p}_{\tilde{\alpha}}^2 + \mathsf{p}_{\beta_-}^2 + \mathsf{p}_{\beta_+}^2 + U(\alpha, \beta_\pm) + 12\mathsf{p}_{\phi}^2 + 12\sum_{j=1}^{3}\left(\mathsf{p}_{\varphi_j}^2 + e^{2\varphi_j}\mathsf{p}_{\varsigma_j}^2\right) \ , \tag{6.165}$$

where the minisuperspace potential $U(\tilde{\alpha}, \beta_\pm)$ is defined by

$$U(\tilde{\alpha}, \beta_\pm) \equiv -12\pi^2 e^{6\tilde{\alpha}}\ {}^{(3)}R \ . \tag{6.166}$$

Commutation relations follow in the form $[q^X, p^Y] = i\delta^{XY}$, and the Wheeler–DeWitt equation is

$$\begin{aligned}\Bigg[e^{-p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}}e^{p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + U\left(\tilde{\alpha}, \beta_\pm\right) - 12\frac{\partial^2}{\partial\phi^2} \\ -12\sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\varphi_j^2} + e^{2\varphi_j}\frac{\partial^2}{\partial\varsigma_j^2}\right)\Bigg]\Psi = 0 \ .\end{aligned} \tag{6.167}$$

Note 6.10 It should be borne in mind that an ambiguity arises in the operator ordering, which can be accounted for by

$$p_{\tilde{\alpha}}^2 = -e^{-p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}}e^{p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}} \ , \tag{6.168}$$

$$p_X^2 = -\frac{\partial}{\partial q^X}\frac{\partial}{\partial q^X}, \qquad \mu \neq 0 \,, \tag{6.169}$$

where p is a constant and no summation is implied in the second equation.

Similarly to the procedure employed in Sect. 6.1, we can supersymmetrize the vacuum Bianchi class A minisuperspace by solving the Euclidean Hamilton–Jacobi equation

$$U = \mathcal{G}^{XY}\frac{\partial \mathsf{W}}{\partial q^X}\frac{\partial \mathsf{W}}{\partial q^Y} \,, \tag{6.170}$$

and hence introducing the fermionic (partner) variables, e.g., χ^X, $\overline{\chi}^Y$. The vacuum Hamiltonian is then equivalent to

$$0 = \mathcal{H}_c \propto \mathcal{G}^{XY} p_X p_Y + U(q^X) \longrightarrow \mathcal{H} \propto \mathcal{S}\overline{\mathcal{S}} + \overline{\mathcal{S}}\mathcal{S} \,, \tag{6.171}$$

with the SUSY constraints

$$\begin{aligned} \mathcal{S} &\sim \chi^X \left(\mathsf{p}_X + \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right) \,, \\ \overline{\mathcal{S}} &\sim \overline{\chi}^X \left(\mathsf{p}_X - \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right) \,, \end{aligned} \tag{6.172}$$

satisfying the condition

$$\mathcal{S}^2 = 0 = \overline{\mathcal{S}}^2 \tag{6.173}$$

(as 'square roots' of the determined Wheeler–DeWitt equation (6.167) in the absence of a matter sector), such that

$$\mathcal{S}\Psi = 0 = \overline{\mathcal{S}}\Psi \,. \tag{6.174}$$

The fermionic filled and purely bosonic components of the supersymmetric wave function are

$$\Psi_{\text{bosonic}} = \mathrm{e}^{-\mathsf{W}} \,, \qquad \Psi_{\text{filled}} = \mathrm{e}^{+\mathsf{W}} \,, \tag{6.175}$$

where the Hamilton–Jacobi equation (6.170) provides

$$\mathsf{W} = \pm\frac{1}{6}m^{ij}\tilde{h}_{ij} \,, \tag{6.176}$$

making (6.175) a solution to the full Wheeler–DeWitt equation (6.167) when *no* matter is included. With the ansatz

$$\Psi = \mathcal{X}(\tilde{\alpha}, \beta_\pm)\mathcal{Y}(\phi, \varphi_j, \varsigma_j) ,$$

and

$$\mathcal{X} = \mathcal{W}(\tilde{\alpha})e^{-\mathsf{W}(\tilde{\alpha}, \beta_\pm)} , \tag{6.177}$$

the Wheeler–DeWitt equation separates into matter and gravitational sectors. More precisely:

- We have

$$\left(\frac{\partial^2}{\partial\tilde{\alpha}^2} + p\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + U - z^2\right)\mathcal{X} = 0 , \tag{6.178}$$

$$\left[\frac{\partial^2}{\partial\phi^2} + \sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\varphi_j^2} + e^{2\varphi_j}\frac{\partial^2}{\partial\varsigma_j^2}\right) - \frac{z^2}{12}\right]\mathcal{Y} = 0 , \tag{6.179}$$

  where z is an arbitrary, possibly imaginary, separation constant that can be interpreted physically as the total momentum eigenvalue of the matter sector.
- The superpotential satisfies the equation

$$\left(\frac{\partial\mathsf{W}}{\partial\tilde{\alpha}}\right)^2 - \left(\frac{\partial\mathsf{W}}{\partial\beta_+}\right)^2 - \left(\frac{\partial\mathsf{W}}{\partial\beta_-}\right)^2 + U = 0 , \tag{6.180}$$

  where (6.180) is the Euclidean Hamilton–Jacobi equation. The bosonic and fermionic filled components of the wave function will be a linear superposition of waves of the form

$$\Psi \sim e^{-\mathsf{W}/\hbar} . \tag{6.181}$$

- A solution for the gravitational component of the wave function is given by [49–52]

$$\mathcal{X} = e^{(3-p/2)\tilde{\alpha}-\mathsf{W}} , \tag{6.182}$$

  provided that we choose a specific factor ordering

$$p^2 = 4(9 - z^2) . \tag{6.183}$$

- For the matter sector, we make the separable ansatz

$$\mathcal{Y} = \mathsf{A}_1(\varphi_1)\mathsf{A}_2(\varphi_2)\mathsf{A}_3(\varphi_3)e^{\pm i\gamma\phi \pm i(\omega_1\varsigma_1 + \omega_2\varsigma_2 + \omega_3\varsigma_3)} , \tag{6.184}$$

where $\{\gamma, \omega_j\}$ are arbitrary constants and $\mathsf{A}_j(\varphi_j)$ are arbitrary functions [49–52], each satisfying the wave equation of Liouville quantum mechanics:

$$\frac{d^2\mathsf{A}_j}{\mathrm{d}\varphi_j^2} - \left(\omega_j^2 \mathrm{e}^{2\varphi_j} + \ell^2 \cos^2\theta_j\right)\mathsf{A}_j = 0 \,, \qquad j = 1, 2, 3 \,, \tag{6.185}$$

where $\ell^2 \equiv \gamma^2 + z^2/12$ and the set of constants $\{\theta_j\}$ satisfy the integrability condition

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1 \,. \tag{6.186}$$

By combining these results for $\mathcal{X}$ and $\mathcal{Y}$, we therefore arrive at

$$\Psi_{\text{bosonic}} \sim \exp\left\{\left(3 - \frac{p}{2}\right)\alpha - \frac{1}{6}\mathrm{e}^{-\phi}\left|m^{ab}h_{ab}\right| + \left[\frac{1}{4}(p-6) \pm \mathrm{i}\gamma\right]\phi\right\} \times \prod_{j=1}^{3}\left[Z_{\pm\lambda\cos\theta_j}\left(\omega_j \mathrm{e}^{\varphi_j}\right)\mathrm{e}^{\pm i\omega_j\varsigma_j}\right] , \tag{6.187}$$

where we have transformed back into the 'original' frame.

One-Fermion States and Beyond (Bianchi I)

In this section, the issue of dualities will be further considered, progressing into the analysis of the one-fermion (and other) sectors, and hence going beyond solutions of the form $\mathrm{e}^{\pm \mathsf{W}}$ previously found. The reader may be thinking that the content will not be so relevant, but this is not the case. We therefore refer to the words of incentive at the beginning of Sect. 6.2. The aim here is to acquire a vast and differentiated experience of the many particularities within SQC, including what has been achieved and how, and also where new areas of research can be found.

We start by rewriting the NS–NS sector of the $(D+1)$-dimensional,[16] tree-level string effective action in (6.151) (see Sects. 6.2.1 and 6.2.3) in the form

$$S \sim \frac{1}{2\lambda_s^{D-1}} \int \mathrm{d}^{D+1}x\sqrt{|g|}\ \mathrm{e}^{-\phi}\left[R - (\nabla\phi)^2 - \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} - V\right] , \tag{6.188}$$

where the Yang–Mills fields are henceforth assumed to be *trivial*, ϕ is the dilaton field, $V(\phi)$ denotes an interaction potential, R is the Ricci curvature scalar of the spacetime with metric g and signature $(-, +, +, \ldots, +)$, $H_{\alpha\beta\gamma} \equiv \partial_{[\alpha}B_{\beta\gamma]}$ is the field strength of the antisymmetric two-form potential $B_{\beta\gamma}$, and $\lambda_s \equiv (\alpha')^{1/2}$ is the fundamental string length scale.

[16] We now take D as generic, not necessarily $D = 3$, as mentioned in Sect. 6.2.1.

For simplicity, we further restrict to a spatially closed, flat, homogeneous (Bianchi type I) spacetime, where the dilaton and 2-form potential are constant on the surfaces of homogeneity $t = \text{constant}$. In addition, $g_{00} = -1$ and $g_{0i} = B_{0i} = 0$. Integrating over the spatial variables in (6.188) then implies [6] that (see (6.159), (6.188))

$$S \simeq \int \mathrm{d}\tau \left\{ \overline{\phi}'^2 + \frac{1}{8}\mathrm{Tr}\left[\mathsf{M}'(\mathsf{M}^{-1})'\right] + V\mathrm{e}^{-2\overline{\phi}} \right\} , \tag{6.189}$$

where again

$$\overline{\phi} \equiv \phi - \frac{1}{2}\ln|g| \tag{6.190}$$

is the shifted dilaton field,

$$\tau \equiv \int^t \mathrm{d}t_1 \mathrm{e}^{\overline{\Phi}(t_1)} \tag{6.191}$$

is the dilaton time parameter, a prime denotes differentiation with respect to τ, and we have specified $\lambda_s \equiv 2$.

It should be stressed that the kinetic sector of action (6.189) is invariant under

$$\tilde{\overline{\phi}} = \overline{\phi} , \qquad \tilde{\mathsf{M}} = \tilde{\Omega}^T \mathsf{M} \tilde{\Omega} , \qquad \tilde{\Omega}^T \check{\eta} \tilde{\Omega} = \check{\eta} , \tag{6.192}$$

where $\tilde{\Omega}$ is a constant matrix. Since the shifted dilaton field transforms as a singlet, this symmetry is respected when V is an arbitrary function of $\overline{\phi}$.

The classical Hamiltonian is then given by (see (6.165))

$$\mathcal{H}_c = \frac{1}{4}\pi_{\overline{\phi}}^2 - 2\mathrm{Tr}\left[\mathsf{M}\pi_{\mathsf{M}}\mathsf{M}\pi_{\mathsf{M}}\right] - V\mathrm{e}^{-2\overline{\phi}} , \tag{6.193}$$

where

$$\pi_{\overline{\phi}} = 2\overline{\phi}' , \qquad \pi_{\mathsf{M}} = -\frac{1}{4}\mathsf{M}^{-1}\mathsf{M}'\mathsf{M}^{-1} ,$$

are the momenta conjugate to $\overline{\phi}$ and M, respectively. The equations of motion for the matrix M yield

$$\mathsf{M}\check{\eta}\mathsf{M}' = C , \tag{6.194}$$

where C is a constant $2D \times 2D$ matrix satisfying the conditions

$$C^T = -C , \qquad \mathsf{M}\check{\eta}C = -C\check{\eta}\mathsf{M} . \tag{6.195}$$

Note 6.11 This constitutes a conservation law in the form

$$\mathsf{M}\pi_{\mathsf{M}} = -\frac{1}{4}C\check{\eta} \, . \tag{6.196}$$

Identifying the momenta with the differential operators

$$\pi_{\overline{\phi}} = -\mathrm{i}\frac{\partial}{\partial\overline{\phi}} \, , \qquad \pi_{\mathsf{M}} = -\mathrm{i}\frac{\partial}{\partial\mathsf{M}} \, , \tag{6.197}$$

and substituting (6.197) into (6.193) then leads to the Wheeler–DeWitt equation (see (6.167)) [6]

$$\left[\frac{\partial^2}{\partial\overline{\phi}^2} + 8\mathrm{Tr}\left(\eta\frac{\partial}{\partial\mathsf{M}}\check{\eta}\frac{\partial}{\partial\mathsf{M}}\right) + 4V\mathrm{e}^{-2\overline{\phi}}\right]\Psi(\overline{\phi},\mathsf{M}) = 0 \, . \tag{6.198}$$

At the quantum level, the wave function should also comply with the first-order constraint

$$\mathrm{i}\mathsf{M}\frac{\partial\Psi}{\partial\mathsf{M}} = \frac{1}{4}C\check{\eta}\Psi \, . \tag{6.199}$$

It implies that the wave function in (6.198) can be separated by specifying

$$\Psi(\mathsf{M},\overline{\phi}) = \mathcal{X}(\mathsf{M})\mathcal{Y}(\overline{\phi}) \, ,$$

where $\mathcal{X}(\mathsf{M})$ and $\mathcal{Y}(\overline{\phi})$ are functions of M and $\overline{\phi}$, respectively. The Wheeler–DeWitt equation then simplifies to:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\overline{\phi}^2} + \wp^2 + 4V(\overline{\phi})\mathrm{e}^{-2\overline{\phi}}\right]\mathcal{Y}(\overline{\phi}) = 0 \, , \tag{6.200}$$

where

$$\wp^2 \equiv \frac{1}{2}\mathrm{Tr}\left(C\check{\eta}\right)^2 \tag{6.201}$$

is a constant.

The one-fermion states are just a step or so ahead. An important ingredient is to employ a *superfield* expansion (see Sect. 3.3). In summary, for superfields including

$$\left\{M_{ij}(\tau), \phi, \overline{\phi}; \left[\psi_{ij}, \overline{\psi}_{ij}, \chi, \overline{\chi}\right]\right\} \qquad (i, j) = (1, 2, \ldots, 2D) \, ,$$

an integration over Grassmann variables leads to an $N = 2$ SUSY action of the form

$$I_{\mathrm{SUSY}} = \int \mathrm{d}\tau \Bigg[\frac{1}{8} \left(\mathrm{i}\psi_{ij}\check{\eta}^{jk}\overline{\psi}'_{kl}\check{\eta}^{li} - \mathrm{i}\psi'_{ij}\check{\eta}^{jk}\overline{\psi}_{kl}\check{\eta}^{li} + \mathsf{M}'_{ij}\check{\eta}^{jk}\mathsf{M}'_{kl}\check{\eta}^{li} \right) \tag{6.202}$$
$$+ (\overline{\phi}')^2 + \frac{\mathrm{i}}{2}(\overline{\chi}\chi' - \overline{\chi}'\chi) - \frac{1}{4}(\partial_{\overline{\phi}}\mathsf{W})^2 - \frac{1}{4}(\partial^2_{\overline{\phi}}\mathsf{W})\,\{\overline{\chi}, \chi\} \Bigg] .$$

In the limit where the Grassmann variables vanish, this reduces to the bosonic action (6.189), if we identify the potential W by means of

$$\left(\partial_{\overline{\phi}}\mathsf{W}\right)^2 = -4V(\overline{\phi})\mathrm{e}^{-2\overline{\phi}} . \tag{6.203}$$

The Hamiltonian derived from the action (6.202) is given by

$$\mathcal{H} = 2\pi_{ij}\check{\eta}^{jk}\pi_{kl}\check{\eta}^{li} + \frac{1}{4}\pi^2_{\overline{\phi}} + \frac{1}{4}\left(\partial_{\overline{\phi}}\mathsf{W}\right)^2 + \frac{1}{4}\left(\partial^2_{\overline{\phi}}\mathsf{W}\right)\{\overline{\chi}, \chi\} , \tag{6.204}$$

where the anticommuting property of the Grassmann variables has been employed, and $\pi_{ij} \equiv \pi_{\mathsf{M}_{ij}}$. The bosonic component of the Hamiltonian (6.204) corresponds to (6.193). The model is quantized by assuming the standard operator realizations for the bosonic variables in (6.197), and further, for the set of Grassmann variables $\{\zeta_{ij}, \theta\}$:

$$\psi_{kl} = \check{\eta}_{kp}\frac{\partial}{\partial\zeta_{pr}}\check{\eta}_{rl} , \qquad \overline{\psi}_{ij} = \zeta_{ij} , \chi = \frac{\partial}{\partial\theta} , \qquad \overline{\chi} = \theta , \tag{6.205}$$

from which the SUSY constraints follow:

$$\mathcal{S} \equiv 2\pi_{ij}\check{\eta}^{jk}\psi_{kl}\check{\eta}^{li} + \frac{1}{\sqrt{2}}\left(\pi_{\overline{\phi}} + \mathrm{i}\partial_{\overline{\phi}}\mathsf{W}\right)\chi , \tag{6.206}$$

$$\overline{\mathcal{S}} \equiv 2\pi_{mn}\check{\eta}^{nr}\overline{\psi}_{rp}\check{\eta}^{pm} + \frac{1}{\sqrt{2}}\left(\pi_{\overline{\phi}} - \mathrm{i}\partial_{\overline{\phi}}\mathsf{W}\right)\overline{\chi} , \tag{6.207}$$

where $\mathcal{S}$ is a (non-Hermitian) linear operator and $\overline{\mathcal{S}}$ is its adjoint. In particular,

$$|\Psi_0\rangle = \mathrm{e}^{-\mathsf{W}(\overline{\phi})}|0\rangle \tag{6.208}$$

is the state with zero fermion number, also written as $|\Psi_0\rangle \equiv \varpi(M_{ij}, \overline{\phi})|0\rangle$, where ϖ is an arbitrary function, and using

$$\frac{\delta\varpi}{\delta\mathsf{M}_{ij}} = 0 , \qquad \left(\frac{\partial}{\partial\overline{\phi}} + \frac{\mathrm{d}\mathsf{W}}{\mathrm{d}\overline{\phi}}\right)\varpi = 0 . \tag{6.209}$$

Let us proceed to find the form of the one-fermion state $|\psi_1\rangle$ [6]. A general ansatz for this component of the wave function is given in terms of the fermion vacuum by

$$|\Psi_1\rangle = \left(f_{ij}\check{\eta}^{jk}\overline{\psi}_{kl}\check{\eta}^{li} + f_\chi \overline{\chi} \right) \mathrm{e}^{-\mathsf{W}}|0\rangle \ , \tag{6.210}$$

where f_{ij} and f_χ are arbitrary functions of the bosonic variables over the configuration space. Then

$$|\Psi_1\rangle \equiv \overline{\mathcal{S}} f \mathrm{e}^{-\mathsf{W}(\overline{\phi})}|0\rangle \ . \tag{6.211}$$

Moreover, f is a solution to the differential equation:

$$\left(2\frac{\delta}{\delta \mathsf{M}_{ij}}\check{\eta}_{jk}\frac{\delta}{\delta \mathsf{M}_{kl}}\check{\eta}_{li} + \frac{1}{4}\frac{\partial^2}{\partial \overline{\phi}^2} - \frac{1}{2}\frac{\mathrm{d}\mathsf{W}}{\mathrm{d}\overline{\phi}}\frac{\partial}{\partial \overline{\phi}} \right) f(\mathsf{M}_{ij}, \overline{\phi}) = 0 \ . \tag{6.212}$$

For a separable solution of the form $f \equiv \mathcal{X}(\mathsf{M}_{ij})\mathcal{Y}(\overline{\phi})$, Eq. (6.212) implies that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\overline{\phi}^2} - 2\frac{\mathrm{d}W}{\mathrm{d}\overline{\phi}}\frac{\mathrm{d}}{\mathrm{d}\overline{\phi}} + c^2 \right) \mathcal{Y}(\overline{\phi}) = 0 \tag{6.213}$$

and

$$\left(\frac{\partial}{\delta \mathsf{M}_{ij}}\check{\eta}_{jk}\frac{\partial}{\delta \mathsf{M}_{kl}}\check{\eta}_{li} - \frac{c^2}{8} \right) \mathcal{X}(\mathsf{M}_{ij}) = 0 \ , \tag{6.214}$$

where c is a separation constant. In the special case where $c = 0$, (6.213) may be solved exactly:

$$\mathcal{Y} = \int^{\overline{\phi}} \mathrm{d}\overline{\phi}_1 \mathrm{e}^{2\mathsf{W}(\overline{\phi}_1)} \ , \tag{6.215}$$

for an arbitrary potential $\mathsf{W}(\overline{\phi})$.

Type II String Theory and S-Duality

As the reader may have noticed, it is rather difficult to go further without additional specific forms of the matrix M. A simple scenario is, from the setting above, to include an $\mathrm{SL}(2,\mathbb{R})$ matrix, corresponding to a subgroup of $\mathrm{O}(d,d)$. The group $\mathrm{SL}(2,\mathbb{R})$ appears in the context of the S-duality group and is useful in the context of type IIB string cosmology.[17] An $\mathrm{SL}(2,\mathbb{R})$ matrix can be introduced with unit determinant as

[17] $\mathrm{SL}(2,\mathbb{R})$ is also possible as a subgroup of the T-duality group.

$$\mathsf{T} \equiv \begin{pmatrix} e^{q\Delta} + e^{-q\Delta}\mathcal{P}^2 & e^{-q\Delta}\mathcal{P} \\ e^{-q\Delta}\mathcal{P} & e^{-q\Delta} \end{pmatrix} , \tag{6.216}$$

where q is a constant. The two scalar *moduli fields* Δ and $\mathcal{P}$ parametrise the coset $SL(2,\mathbb{R})/\,SO(2)$.

The action (6.189) can be written in an $SL(2,\mathbb{R})$ invariant form by replacing the kinetic term for the M matrix by $\mathrm{Tr}[\mathsf{T}'(\mathsf{T}^{-1})']/4$. The classical equations of motion for the moduli fields then imply that the momentum conjugate to $\mathcal{P}$, viz., $\pi_{\mathcal{P}} = -\mathcal{P}' \exp(-2q\Delta)$, is conserved. Thus, the quantum constraint (6.199) takes the simple form $\mathrm{i}\partial\Psi/\partial\mathcal{P} = L_{\mathcal{P}}\Psi$, where $L_{\mathcal{P}}$ is an arbitrary constant. For separable solutions, $\Psi \equiv \mathcal{X}(\mathsf{T})\mathcal{Y}(\overline{\phi})$, the $\mathcal{X}(\mathsf{T})$ component of the wave function can then be evaluated by separating the corresponding Wheeler–DeWitt equation (6.198). It follows that

$$\left(\frac{1}{2q^2}\frac{\partial^2}{\partial\Delta^2} + \frac{1}{2}e^{2q\Delta}\frac{\partial^2}{\partial\mathcal{P}^2} + B^2\right)\mathcal{X}(\mathsf{T}) = 0 , \tag{6.217}$$

and the general solution to this equation that is consistent with the first-order momentum constraint (6.199) is given by

$$\mathcal{X} = e^{-\mathrm{i}L_{\mathcal{P}}\mathcal{P}} Z_{\sqrt{2}B}\left(L_{\mathcal{P}}e^{q\Delta}\right) , \tag{6.218}$$

where $Z_{\sqrt{2}B}$ is a linear combination of modified Bessel functions of order $\sqrt{2}B$.

The supersymmetric extension can then be carried out for this model by defining a *superfield* $N_{ij} \equiv T_{ij} + \mathrm{i}\overline{A}_{ij}\theta + \mathrm{i}A_{ij}\overline{\theta} + C_{ij}\theta\overline{\theta}$. Similar conclusions therefore apply and, in particular, the structure of (6.214) for the one-fermion state is formally equivalent. The solutions are given in terms of modified Bessel functions.

Note 6.12 When $D = 3$, (6.188) also exhibits an $SL(2,\mathbb{R})$ S-duality. In four dimensions, the 3-form field strength of the NS–NS two-form potential is dual to a one-form corresponding to the field strength of a pseudoscalar *axion* field σ :

$$H^{\alpha\beta\gamma} \equiv e^{\phi}\varepsilon^{\alpha\beta\gamma\delta}\nabla_{\delta}\sigma , \tag{6.219}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the covariantly constant antisymmetric 4-form. Performing the conformal transformation

$$\tilde{g}_{\mu\nu} \equiv \tilde{\Omega}^2 g_{\mu\nu} , \qquad \tilde{\Omega}^2 \equiv e^{-\phi} , \tag{6.220}$$

we find that the action (6.188) transforms to

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}\left(\tilde{\nabla}\phi\right)^2 - \frac{1}{2}e^{2\phi}\left(\tilde{\nabla}\sigma\right)^2\right] . \tag{6.221}$$

The dilaton and axion fields parametrize the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset. A supersymmetric extension of the FRW cosmologies may also be developed for this model [6].

6.2.2 P-Brane Backgrounds

The methodology employed in this section follows the discussion in Sect. 6.1.3, but the physical (and therefore, cosmological) context is quite 'stringy'. In fact, developments in superstring theory suggest that, in a Planck length regime, the quantum fluctuations are very large so that the coupling may increase, and consequently the string degrees of freedom would *not* be the relevant ones. Instead, soliton degrees of freedom such as D(irichlet) p-branes would become more important as the strong coupling regime becomes dominant. In particular, the quantum fluctuations will be strongly influenced by the effects of those p-branes. But then what would be the effect of these new physical degrees of freedom on, say, the very early universe, in particular from a quantum cosmological point of view?

Within the SUSY context described in this chapter [8], we will apply it to retrieve a set of quantum states associated with a specific FRW cosmological model. To obtain an adequate low energy effective theory with gravity and solitons, actions of Born–Infeld type have been the subject of wide interest in string/M-theory [57].

Here we shall employ instead the useful result (see [8] and references therein) that the natural metric that couples to a p-brane is the Einstein metric multiplied by a certain power of the dilaton field: a particular Brans–Dicke action is obtained [58]. It possesses a very specific coupling $\omega(d)$, depending on the dimension d of the world-volume swept out by the p-brane solution in the physical spacetime. This overall Brans–Dicke scenario represents a gas of solitonic p-branes, including a perfect fluid matter corresponding to form fields associated with the branes and originating from the appropriate compactification of the ten-dimensional low energy effective theory. This determines an arbitrary relation $P = \gamma\rho$ for the pressure and energy densities in this FRW background. This matter content can either be present in a Ramond/Ramond (R/R) or in a Neveu–Schwarz/Neveu–Schwarz (NS/NS) sector. In the former the perfect fluid content is not coupled to the dilaton, while in the latter it is.

We take the following actions to describe FRW minisuperspaces extracted from a D(irichlet) p-brane framework. For the R/R case we employ

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{\mathrm{e}^{-\phi}\big[R - \omega(\nabla\phi)^2\big] + L_{\mathrm{m}}\Big\}\ , \tag{6.222}$$

and for the NS/NS case we use

$$S = \int d^4x \sqrt{-g} \left\{ e^{-\phi} \left[R - \omega(\nabla\phi)^2 + L_m \right] \right\} , \tag{6.223}$$

where

$$\omega \equiv -\frac{4(p-1) - (p+1)^2}{2(p-1) - (p+1)^2} , \tag{6.224}$$

for $p = d - 1$, e.g., a particle is a 0-brane, with $d = 1$ and $p = 0$, whereas a string is a 1-brane with $d = 2$ and $p = 1$. We further assume a flat FRW geometry with metric given by the expression

$$ds^2 = -\mathcal{N}^2 dt^2 + e^{2\alpha} d\mathbf{x}^2 , \tag{6.225}$$

and a perfect fluid matter content for L_m, satisfying $\rho = \rho_0 e^{-3(1+\gamma)\alpha}$. In addition, we will employ a *new* time variable $dt \equiv d\tau e^{3\alpha-\phi}$ and further define the following new parameters and variables $X(a,\phi)$, $Y(a,\phi)$:

- For the R/R case:

$$\kappa_{R/R} = 3(1-\gamma)^2(\omega - \omega_{\kappa_{R/R}}) , \tag{6.226}$$

$$\mu_{R/R} = -\frac{8}{\kappa_{R/R}}(\omega - \omega_{\mu_{R/R}}) , \tag{6.227}$$

$$\nu_{R/R} = -2(1-\gamma)(\omega - \omega_{\nu_{R/R}}) , \tag{6.228}$$

$$\omega_{\kappa_{R/R}} = -\frac{4 - 6\gamma}{3(1-\gamma)^2} , \tag{6.229}$$

$$\omega_{\mu_{R/R}} = -\frac{3}{2} , \tag{6.230}$$

$$\omega_{\nu_{R/R}} = -\frac{1}{1-\gamma} , \tag{6.231}$$

$$-2X = 3(1-\gamma)\alpha - \phi , \tag{6.232}$$

$$Y = \alpha + \frac{\nu_{R/R}}{\kappa_{R/R}} X . \tag{6.233}$$

- For the NS/NS case:

$$\kappa_{NS} = \frac{9}{4}(1-\gamma)^2(\omega - \omega_{\kappa_{NS}}) , \tag{6.234}$$

$$\mu_{NS} = -\frac{3(2\omega + 3)}{\kappa_{NS}} , \tag{6.235}$$

$$\nu_{NS} = 3(1-\gamma)(\omega - \omega_{\nu_{NS}}) , \tag{6.236}$$

$$\omega_{\kappa_{\rm NS}} = \frac{4}{3}\frac{3\gamma - 1}{(1-\gamma)^2} \ , \tag{6.237}$$

$$\omega_{\nu_{\rm NS}} = -\frac{2}{1-\gamma} \ , \tag{6.238}$$

$$-2X = 3(1-\gamma)\alpha - 2\phi \ , \tag{6.239}$$

$$Y = \alpha + \frac{\nu_{\rm NS}}{2\kappa_{\rm NS}} X \ . \tag{6.240}$$

From these we then write the actions (6.222) and (6.223) in the same formal manner as the following reduced actions, using primes to denote differentiation with respect to τ:

$$S^{\rm R/R} = \int {\rm d}\tau \left[\frac{1}{\mathcal{N}^2} \left(3\kappa_{\rm R/R} Y'^2 + \mu_{\rm R/R} X'^2 \right) - \mathcal{N}^2 \rho {\rm e}^{-2X} \right] \tag{6.241}$$

and

$$S^{\rm NS} = \int {\rm d}\tau \left[\frac{1}{\mathcal{N}^2} \left(3\kappa_{\rm NS} Y'^2 + \mu_{\rm NS} X'^2 \right) - \mathcal{N}^2 \rho {\rm e}^{-2X} \right] \ . \tag{6.242}$$

The corresponding canonical Hamiltonians are, after a suitable variable redefinition (up to constant factors),

$$\varepsilon_{\kappa_{\rm R/R}} \pi_Y^2 + \varepsilon_{\mu_{\rm R/R}} \pi_X^2 + 4\rho_0 {\rm e}^{-2X/m_{\rm R/R}^{1/2}} = 0 \tag{6.243}$$

and

$$\varepsilon_{\kappa_{\rm NS}} \pi_Y^2 + \varepsilon_{\mu_{\rm NS}} \pi_X^2 + 4\rho_0 {\rm e}^{-2X/m_{\rm NS}^{1/2}} = 0 \ . \tag{6.244}$$

In obtaining the above Hamiltonian expressions, we used the following procedure. For the R/R case, e.g., we rewrite $3\kappa_{\rm R/R} Y'^2 \equiv \overline{Y'}^2$ and put

$$\kappa_{\rm R/R} = \left|\kappa_{\rm R/R}\right| \varepsilon_{\kappa_{\rm R/R}} \equiv k_{\rm R/R} \varepsilon_{\kappa_{\rm R/R}} \ ,$$

where $k_{\rm R/R} > 0$ and $\varepsilon_{\kappa_{\rm R/R}} \equiv {\rm sign}\ \kappa_{\rm R/R}$. Then we drop the bar over all the variables. Accordingly, we have used the notation:

$$\begin{aligned} &\kappa_{\rm R/R} \equiv k_{\rm R/R} \varepsilon_{\kappa_{\rm R/R}} \ , && \varepsilon_{\kappa_{\rm R/R}} \equiv {\rm sign}\ \kappa_{\rm R/R} \equiv \pm 1 \ , \\ &\kappa_{\rm NS} \equiv k_{\rm NS} \varepsilon_{\kappa_{\rm NS}} \ , && \varepsilon_{\kappa_{\rm NS}} \equiv {\rm sign}\ \kappa_{\rm NS} = \pm 1 \ , \\ &\mu_{\rm R/R} \equiv m_{\rm R/R} \varepsilon_{\mu_{\rm R/R}} \ , && \varepsilon_{\mu_{\rm R/R}} \equiv {\rm sign}\ \mu_{\rm R/R} \equiv \pm 1 \ , \\ &\mu_{\rm NS} = m_{\rm NS} \varepsilon_{\mu_{\rm NS}} \ , && \varepsilon_{\mu_{\rm NS}} \equiv {\rm sign}\ \mu_{\rm NS} = \pm 1 \ . \end{aligned} \tag{6.245}$$

The corresponding Wheeler–DeWitt equations are thus

$$-\varepsilon_{\kappa_{\mathrm{R/R}}} \frac{\partial^2 \Psi_{\mathrm{R/R}}}{\partial Y^2} - \varepsilon_{\mu_{\mathrm{R/R}}} \frac{\partial^2 \Psi_{\mathrm{R/R}}}{\partial X^2} + 4\rho_0 \mathrm{e}^{-2X/m_{\mathrm{R/R}}^{1/2}} \Psi_{\mathrm{R/R}} = 0 \tag{6.246}$$

and

$$-\varepsilon_{\kappa_{\mathrm{NS}}} \frac{\partial^2 \Psi_{\mathrm{NS}}}{\partial Y^2} - \varepsilon_{\mu_{\mathrm{NS}}} \frac{\partial^2 \Psi_{\mathrm{NS}}}{\partial X^2} + 4\rho_0 \mathrm{e}^{-2X/m_{\mathrm{NS}}^{1/2}} \Psi_{\mathrm{NS}} = 0 \, . \tag{6.247}$$

Let us rewrite both in a common formal expression as

$$\left(-s_1 \frac{\partial^2}{\partial y^2} - s_2 \frac{\partial^2}{\partial x^2} + 4\rho_o \mathrm{e}^{-2x/M} \right) \Psi = 0 \, , \tag{6.248}$$

where $s_1 \equiv \varepsilon_{\kappa_{\mathrm{R/R}}}$ or $\varepsilon_{\kappa_{\mathrm{NS}}}$, $s_2 \equiv \varepsilon_{\mu_{\mathrm{R/R}}}$ or $\varepsilon_{\mu_{\mathrm{NS}}}$, and here $M \equiv \sqrt{m_{\mathrm{R/R}}}$ or $\sqrt{m_{\mathrm{NS}}}$, $y \equiv Y$, $x \equiv X$.

Concerning $N = 2$ SUSY in these p-brane FRW settings, the spatially flat and isotropic Brans–Dicke cosmology exhibits a discrete scale factor duality. This is an important feature to explore within the above quantum cosmological model.

In the variables X, Y, we have the following duality for the scenario induced by D p-brane degrees of freedom:

$$X = \overline{X} \, , \qquad Y = -\overline{Y} \, . \tag{6.249}$$

In the presence of this duality, one may check for wave function solutions with $N = 2$ SUSY quantum cosmology. Basically, from the Wheeler–DeWitt equation

$$\left(-s_1 \frac{\partial^2}{\partial y^2} - s_2 \frac{\partial^2}{\partial x^2} + 4\rho_0 \mathrm{e}^{-2x/M} \right) \Psi = 0 \, , \tag{6.250}$$

one needs to consider the corresponding Euclidean Hamilton–Jacobi equation

$$\mathcal{G}^{XY} \frac{\partial \mathsf{W}}{\partial q^X} \frac{\partial \mathsf{W}}{\partial q^Y} = U(q^X) \, , \tag{6.251}$$

with

$$\mathcal{G}^{XY} \equiv (s_1, s_2) \, , \qquad q^X = (y, x) = (q^0, q^1) \, , \tag{6.252}$$

and

$$U = 4\rho_0 \mathrm{e}^{-2x/M} \, , \tag{6.253}$$

i.e.,

$$s_1 \left(\frac{\partial \mathsf{W}}{\partial y} \right)^2 + s_2 \left(\frac{\partial \mathsf{W}}{\partial x} \right)^2 = U(x) = 4\rho_0 \mathrm{e}^{-2x/M} \, . \tag{6.254}$$

We seek solutions to the above equation. One possibility is

$$\mathsf{W} = \mathsf{J}(x) + cA\sqrt{y^2}\ . \tag{6.255}$$

For c, A constants, we find that it reduces to

$$s_2\left(\frac{\partial \mathsf{J}}{\partial x}\right)^2 = 4\rho_0 \mathrm{e}^{-2x/M} - s_1 A^2\ . \tag{6.256}$$

Solutions for (6.256), and hence (6.254), will be determined by the choices of whether s_1 and/or s_2 are ± 1 or zero, whether $\rho_0 > 0$ or $\rho_0 < 0$, and moreover whether $4\rho_0 \mathrm{e}^{-2x/M}$ is positive, negative, or zero. We can make an analysis according to 4 situations:

- Case 1: $s_2 = 1, s_1 = 1$.
- Case 2: $s_2 = -1, s_1 = 1$.
- Case 3: $s_2 = -1, s_1 = -1$.
- Case 4: $s_2 = 1, s_1 = -1$.

In each we can investigate what the values/sign of ρ_0 and the term $4\rho_0 \mathrm{e}^{-2x/M} - s_1 A^2$ determine. The solutions for $\mathsf{J}(x)$ do exist, and satisfy the duality properties.

Hence, we can have an $N = 2$ supersymmetric extension for our minisuperspace model. Let us take the case $A = 0$ for simplicity, where there are two cases for

$$s_2\left(\frac{\partial \mathsf{J}}{\partial x}\right)^2 = 4\rho_0 \mathrm{e}^{-2x/M}\ , \tag{6.257}$$

according to $s_2 = -1, \rho_0 < 0$, i.e., $\widehat{\rho_0} > 0$, or $s_2 = 1, \rho_0 > 0$. The solutions are [8]

$$\mathsf{J} = \mp 2\sqrt{\rho_0} M \mathrm{e}^{-x/M}\ , \tag{6.258}$$

or

$$\mathsf{J} = \mp 2\sqrt{\widehat{\rho_0}} M \mathrm{e}^{-x/M}\ . \tag{6.259}$$

We write the $N = 2$ SUSY constraints in the form

$$\mathcal{S} = \psi^X \left(\pi_X + \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right)\ , \tag{6.260}$$

$$\overline{\mathcal{S}} = \overline{\psi}^X \left(\pi_X - \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right)\ , \tag{6.261}$$

where $\overline{\psi}^X, \psi^Y = G^{XY}\partial/\partial\theta^X$ are Grassmanian variables, and the ansatz for the $N = 2$ SUSY wave function of the universe is

$$\Psi_{N=2} = \mathsf{A}_+ + \mathsf{B}_X\theta^X + \mathsf{A}_-\theta^0\theta^1 \ , \tag{6.262}$$

with $\mathsf{B}_X\theta^X = \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1$. Now we apply

$$\overline{\mathcal{S}}\Psi_{N=2} = 0 \ , \tag{6.263}$$

$$\mathcal{S}\Psi_{N=2} = 0 \ , \tag{6.264}$$

to get a set of first-order partial differential equations whose solutions include

$$\mathsf{A}_+ \sim \mathrm{e}^{\mathsf{W}} \ , \tag{6.265}$$

$$\mathsf{A}_- \sim \mathrm{e}^{-\mathsf{W}} \ , \tag{6.266}$$

$$\mathsf{W} \equiv \pm 2\sqrt{\rho_0}M\mathrm{e}^{-x/M} \ . \tag{6.267}$$

Moreover, it can be shown that B_0 and B_1 satisfy the Lagrange equation.

6.2.3 Induced Gravity Scenarios

In this section we will elaborate (see [53]) in a broader manner on the methods introduced in Sect. 6.2 for gravitational theories that *generalize* general relativity and can be related to the gravitational sector of superstring theories. In Sect. 6.2.4, we will then consider the specific case of scalar–tensor (Brans–Dicke) scenarios.

Actions in the following form appear in string theory (to leading order in the inverse string tension α', at the tree-level effective form for the closed bosonic string in two dimensions [58, 56]:

$$S \sim -\frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{-g}\mathrm{e}^{-2\phi}\left[R - 4\left(\nabla\phi\right)^2 + D(\phi)\right] \ , \tag{6.268}$$

where $D(\phi) = c = 16/\alpha'$ is proportional to the effective central charge and the tachyon field is assumed to be zero. The field strength $H_{\mu\nu\lambda}$ of the antisymmetric tensor field vanishes identically in two dimensions. The action (6.268) belongs to the class of actions for 2D dilaton–gravity, invariant under local reparametrizations, and does not contain third or higher order derivatives:

$$S \sim -\frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{-g}\Big[c_2(\phi)R - c_1(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\Big] \ , \tag{6.269}$$

where $g_{\mu\nu}$ is the metric on the two-dimensional spacetime manifold, g is its determinant, $c_1(\phi)$ and $c_2(\phi)$ are functions of the dilaton scalar field ϕ, R is the curvature scalar, and $V(\phi)$ is the dilaton potential.

By choosing particular expressions for the functions c_1, c_2, V, we make them correspond to different two-dimensional models. For example, the induced gravity action is a special case of (6.269) with $c_1 = 1$, $c_2 = 2\phi$, and $V = 4\Lambda^2$. Also of interest is the spherically symmetric 4D Einstein–Hilbert action. This is equivalent to (6.269) if $c_2 = \mathrm{e}^{-2\phi}$, $c_1 = 2c_2$ and $V = 2(1 - \Lambda\mathrm{e}^{-2\phi})$, where Λ is the four-dimensional cosmological constant. In this example the dilaton field is related to the radius of the two-sphere.

The action (6.269) may be further simplified after suitable redefinitions of the dilaton and graviton fields. If c_1 and c_2 are positive-definite functions, we may define a new scalar field

$$\Phi \equiv \sqrt{2}\int \mathrm{d}\phi\sqrt{c_1(\phi)}\ , \tag{6.270}$$

and perform the conformal transformation

$$\tilde{g}_{\mu\nu} \equiv \tilde{\Omega}^2 g_{\mu\nu}\ , \qquad \tilde{\Omega}^2 \equiv \mathrm{e}^{-2\rho}\ , \tag{6.271}$$

where

$$\rho = \frac{c_2}{q^2} - \frac{1}{4}\int^{\Phi} \mathrm{d}\Phi' \left(\frac{\mathrm{d}c_2}{\mathrm{d}\Phi'}\right)^{-1}\ , \tag{6.272}$$

and q is a constant. As a consequence the action (6.269) transforms into

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{-\tilde{g}}\left[\frac{1}{2}q\varphi\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right]\ , \tag{6.273}$$

where

$$q\varphi \equiv 2c_2[\phi(\Phi)]\ , \qquad V(\varphi) = \mathrm{e}^{2\rho}V(\phi)\ . \tag{6.274}$$

Let us then consider FRW cosmologies of the form

$$\mathrm{d}s^2 = -\mathcal{N}^2(t)\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathbf{x}^2\ , \tag{6.275}$$

where as usual $a(t)$ is the radius of the spatial hypersurfaces and $\mathcal{N}(t)$ is the lapse function:

- The action (6.273) now takes the form

$$S = \int \mathrm{d}t\left[\frac{1}{\mathcal{N}}\left(q\dot{\varphi}\dot{a} + \frac{1}{2}a\dot{\varphi}^2\right) - \mathcal{N}aV\right]\ , \tag{6.276}$$

  after integration over the spatial sections.

- Define a new coordinate pair

$$u \equiv \sqrt{2} q a \mathrm{e}^{\varphi/2q} \ , \qquad v \equiv \sqrt{2} q \mathrm{e}^{-\varphi/2q} \ , \tag{6.277}$$

  with $0 < \{a, q\varphi\} < +\infty$ corresponding to the range $0 < u < +\infty$ and $0 < v < \sqrt{2}q$.
- Hence the action (6.276) has the form

$$S = -\int \mathrm{d}t \left(\frac{1}{\mathcal{N}} \dot{u}\dot{v} + \mathcal{N} a V \right) \ . \tag{6.278}$$

The Hamiltonian constraint is then given by

$$\frac{1}{\mathcal{N}^2} \dot{u}\dot{v} - \frac{uv}{2q^2} V[\psi(v)] = 0 \ . \tag{6.279}$$

Let us introduce the rescaled variables

$$\alpha \equiv \frac{u^2}{4q^2} \tag{6.280}$$

and

$$\beta \equiv \int^v \mathrm{d}v' v' V[\varphi(v')] = -q \int^\varphi \mathrm{d}\varphi' \mathrm{e}^{-\varphi'/q} V(\varphi') \ , \tag{6.281}$$

where β represents a rescaled dilaton field, with $0 < \alpha < +\infty$. We further restrict our discussion to potentials that are either positive- or negative-definite, and the action takes the form

$$S = -\int \mathrm{d}t \left(\frac{1}{\mathcal{N} a V} \dot{\alpha}\dot{\beta} + \mathcal{N} a V \right) \ , \tag{6.282}$$

while the corresponding Hamiltonian constraint (6.279) becomes

$$\mathcal{H} = aV(\mathsf{p}_\alpha \mathsf{p}_\beta - 1) = 0 \ , \tag{6.283}$$

where $\mathsf{p}_\alpha = -\dot{\beta}/(\mathcal{N}aV)$ and $\mathsf{p}_\beta = -\dot{\alpha}/(\mathcal{N}aV)$ are the momenta conjugate to α and β, respectively. At this stage [53], we identify the symmetries of the classical Hamiltonian (6.279):

- The kinetic part is invariant under the simultaneous interchanges $u \longleftrightarrow \pm v$.
- The full Hamiltonian is invariant under this interchange in the case where the potential is constant, i.e., if $V \equiv \Lambda^2$.
- This form of the potential in (6.273) corresponds to the induced gravity action for $q = \sqrt{8}$.
- These symmetries are equivalent to an invariance under $\alpha \leftrightarrow \pm\beta$.

From (6.281), it follows that $\beta = q^2\Lambda^2 \mathrm{e}^{-\psi/q}$, and the non-zero components of the minisuperspace metric are $\mathcal{G}_{XY} = a\Lambda^2$. The superpotential is therefore given by $W = -2a\Lambda^2$ and it follows that one solution that respects the symmetries of the Hamiltonian is[18]

$$\mathsf{W} = -2\mathrm{i}(\alpha\beta)^{1/2} \ . \tag{6.284}$$

The functional form of the supersymmetric wave function may now be determined by solving the supersymmetry constraints in the usual form. The general supersymmetric wave function may be expanded as

$$\Psi = \mathsf{A}_+ + \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1 + \mathsf{C}_2\theta^0\theta^1 \ , \tag{6.285}$$

where the bosonic functions $\{\mathsf{A}_+, \mathsf{B}_0, \mathsf{B}_1, \mathsf{C}_2\}$ are functions of $\{\alpha, \beta\}$ alone.

The annihilation of the wave function by the supercharge operators then translates into a set of coupled, first-order partial differential equations (see Exercise 6.7), whose solution is

$$\mathsf{A}_+ = \mathrm{e}^{-\mathsf{W}} \ ,$$

$$\mathsf{C}_2 = \mathrm{e}^{\mathsf{W}} \ , \tag{6.286}$$

$$\mathsf{B}_0 = \frac{\partial \mathsf{F}}{\partial \alpha} + \mathsf{F}\frac{\partial \mathsf{W}}{\partial \alpha} \ , \tag{6.287}$$

$$\mathsf{B}_1 = \frac{\partial \mathsf{F}}{\partial \beta} + \mathsf{F}\frac{\partial \mathsf{W}}{\partial \beta} \ , \tag{6.288}$$

where the function $\mathsf{F} = \mathsf{F}(\alpha, \beta)$ is

$$\mathsf{F} = H_n\left(\sqrt{\gamma\beta} + \sqrt{\alpha}\right) H_{n+1}\left(\sqrt{\gamma\beta} - \sqrt{\alpha}\right) \mathrm{e}^{-(\alpha+\gamma\beta)} \ . \tag{6.289}$$

Note 6.13 The functions A_+ and C_2 represent the empty and filled fermion sectors of the Hilbert space. Both may be interpreted as lowest order, WKB approximations to exact solutions of the corresponding bosonic Wheeler–DeWitt equation. The general form of the associated solution is given by a linear combination of modified Bessel functions $I_0(x)$ and $K_0(x)$, where $x \equiv 2\sqrt{\alpha\gamma\beta}$. The point to note is that, for large x, these functions have the asymptotic forms $I_0 \propto \mathrm{e}^x$ and $K_0 \propto \mathrm{e}^{-x}$, respectively, and these limits correspond to the solutions C_2 and A_+. As we have been suggesting, the supersymmetric vacua are therefore closely related to their semi-classical limits, and correspond to the pure bosonic states of lowest energy.

[18] $\Lambda^2 < 0$ is required for the Euclidean action to be real (see Sects. 6.1 and 6.1.1).

6.2.4 Brans–Dicke Theory

Scalar–tensor gravity theories (see Sects. 6.2.2 and 6.2.3) must be considered when investigating the very early Universe [58, 48, 56]. On the one hand, these theories introduce a non-minimal coupling of a scalar field to the spacetime curvature, extending beyond general relativity in a simple manner. Furthermore, such couplings are possible in the low energy limit of superstring theory. In addition, theories containing higher order terms in the Ricci scalar, related to effective quantum effects, may also be expressed in a scalar–tensor form by means of a suitable conformal transformation. Finally, the dimensional reduction of higher-dimensional gravity may also result in an effective scalar–tensor theory.

As emphasized in the previous section, in the domain of SQC, it is also relevant to explore the reduced (string) actions that exhibit symmetries, and also *scale factor duality*.

Note 6.14 The symmetry group will be Z_2^{D-1}, corresponding to inverting the scale factor and shifting the value of the dilaton (see Sect. 6.2.1). Recall that scale factor duality is a special case of a $\mathrm{O}(d,d)$ symmetry. It relates expanding dimensions to contracting ones. It also constitutes the basis for the pre-big bang cosmological model developed by Gasperini and Veneziano [48].

In this section, we consider, in an SQC context, the D-dimensional vacuum theory [see (6.268)] [5–7]

$$S = \int \mathrm{d}^D x \sqrt{-g} \mathrm{e}^{-\phi} \left[R - \omega (\nabla\phi)^2 - 2\Lambda \right] , \tag{6.290}$$

where ϕ represents the dilaton field, g is the determinant of the spacetime metric $g_{\mu\nu}$, ω is (here) a spacetime constant, and Λ is the effective cosmological constant. This theory represents one of the simplest extensions to Einsteinian gravity and is formally equivalent to the standard Brans–Dicke theory when $\Lambda = 0$.

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The spacetime metric is therefore taken as $g = \mathrm{diag}[-\mathcal{N}(t), \mathsf{h}(t)]$, where t represents cosmic time, $\mathcal{N}(t)$ is the lapse function, and the $(D-1)\times(D-1)$ matrix $\mathsf{h}(t)$ is the metric on the spatial hypersurfaces. The dilaton field is constant on these surfaces and $\omega > -(D-1)/(D-2)$. For the class of spatially flat, isotropic and homogeneous universes, the action simplifies further to

$$S = \int \mathrm{d}t\, \mathrm{e}^{(D-1)\alpha-\phi} \left\{ \frac{1}{\mathcal{N}} \Big[-(D-1)(D-2)\dot{\alpha}^2 + 2(D-1)\dot{\alpha}\dot{\phi} + \omega\dot{\phi}^2 \Big] - 2N\Lambda \right\} , \tag{6.291}$$

where $e^{\alpha(t)}$ represents the scale factor of the universe, a dot denotes differentiation with respect to t, and a boundary term has been neglected. For $D \geq 3$, define

$$x \equiv \exp\left\{\left(\frac{D-1}{2} + \frac{\gamma}{2}\right)\left[\alpha + \frac{1}{D-2}\left(\frac{1}{\gamma} - 1\right)\phi\right]\right\} ,$$

$$y \equiv \exp\left\{\left(\frac{D-1}{2} - \frac{\gamma}{2}\right)\left[\alpha - \frac{1}{D-2}\left(\frac{1}{\gamma} + 1\right)\phi\right]\right\} , \qquad (6.292)$$

where

$$\gamma \equiv \left[\frac{D-1}{D-1+(D-2)\omega}\right]^{1/2} . \qquad (6.293)$$

A second coordinate pair

$$w \equiv \varepsilon^{1/2}\left[\frac{D-1+(D-2)\omega}{D+(D-1)\omega}\right]^{1/2}(x-y) ,$$

$$z \equiv \varepsilon^{1/2}\left[\frac{D-1+(D-2)\omega}{D+(D-1)\omega}\right]^{1/2}(x+y) , \qquad (6.294)$$

where $\varepsilon = +1$ if $\omega > -D/(D-1)$ and $\varepsilon = -1$ if $\omega < -D/(D-1)$, brings the action into the form

$$S = \frac{1}{\varepsilon}\int \mathrm{d}t\left[\frac{1}{\mathcal{N}}\left(\dot{w}^2 - \dot{z}^2\right) - \frac{\Upsilon}{4}\left(w^2 - z^2\right)\mathcal{N}\right] , \qquad (6.295)$$

where

$$\Upsilon \equiv -2\Lambda\left[\frac{D+(D-1)\omega}{D-1+(D-2)\omega}\right] . \qquad (6.296)$$

This is the action for the *constrained oscillator–ghost oscillator* pair when $\Upsilon > 0$ (see Exercise 6.7). The pair oscillate with identical frequency, but have equal and opposite energy. The scale factor duality invariance of the action (6.291) becomes formally equivalent to the simultaneous interchange of the canonical variables $x \leftrightarrow y$.

The classical Hamiltonian for this system can be written as

$$2\mathcal{H}_\mathrm{c} = \mathcal{G}^{XY}\mathsf{p}_X\mathsf{p}_Y + U(q^X) = 0 , \qquad (6.297)$$

where $X, Y = 0, 1$, and $\mathcal{G}^{XY} = \mathrm{diag}[-1/2, 1/2]$, with $q^0 = w$ and $q^1 = z$. The momenta conjugate to these variables are $\mathsf{p}_0 = 2\dot{w}/\mathcal{N}$ and $\mathsf{p}_1 = -2\dot{z}/\mathcal{N}$, respectively, and the potential is given by

$$U = -\Upsilon(w^2 - z^2)/2 . \tag{6.298}$$

The Wheeler–DeWitt equation follows from $\left[q^X, p^Y\right] = \mathrm{i}\delta^{XY}$ with $\mathsf{p}^X = -\mathrm{i}\partial_X$ and we set $\hbar = 1$:

$$\left[\frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial z^2} - \mathsf{p}(w^2 - z^2)\right]\Psi = 0 , \tag{6.299}$$

with solutions of the form

$$\Psi_n = H_n(\Upsilon^{1/4} w) H_n(\Upsilon^{1/4} z) \mathrm{e}^{-\sqrt{\Upsilon}(w^2+z^2)/2} , \tag{6.300}$$

where H_n is the Hermite polynomial of order n. When $\Upsilon > 0$, these solutions form a discrete basis for any bounded wave function $\Psi = \sum c_n \Psi_n$, where c_n are complex coefficients. The ground state corresponds to $n = 0$ and excited states to $n > 0$. These states do not oscillate, and therefore represent Euclidean geometries. Oscillating wave functions representing Lorenztian geometries may be found, however, when the c_n take appropriate values.

There exists a *hidden supersymmetry* if there is a solution to the Euclidean Hamilton–Jacobi equation that respects the same symmetries as $\mathcal{H}_{\mathrm{c}}$. As the Euclidean Hamilton–Jacobi equation has the form

$$\mathcal{G}^{XY} \frac{\partial \mathsf{W}}{\partial q^X} \frac{\partial \mathsf{W}}{\partial q^Y} = U , \tag{6.301}$$

where $\mathsf{W} = \mathsf{W}(q^X)$, it admits the exact solution

$$\mathsf{W} = \sqrt{\Upsilon}(w^2 + z^2)/2 , \tag{6.302}$$

which is indeed invariant under the duality transformation $w \to -w$ and $z \to z$. The supersymmetry constraints are linear operators of the form

$$\mathcal{S} \equiv \psi^X \left(\mathsf{p}_X + \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right), \qquad \overline{\mathcal{S}} \equiv \overline{\psi}^X \left(\mathsf{p}_X - \mathrm{i}\frac{\partial \mathsf{W}}{\partial q^X}\right) , \tag{6.303}$$

where the fermionic degrees of freedom $\psi^X, \overline{\psi}^Y$ satisfy the spinor algebra

$$\left\{\psi^X, \psi^Y\right\} = \left\{\overline{\psi}^X, \overline{\psi}^Y\right\} = 0 , \qquad \left\{\psi^X, \overline{\psi}^Y\right\} = \mathcal{G}^{XY} , \tag{6.304}$$

and the bosonic degrees of freedom satisfy the usual anticommutation relation.[19]

A representation for the fermionic sector in terms of the Grassmann variables $\overline{\psi}^X = \theta^X$ and $\psi^X = \mathcal{G}^{XY}\partial/\partial\theta^Y$ allows the SUSY wave function to be expanded in terms of these variables:

$$\Psi = \mathsf{A}_+ + \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1 + \mathsf{A}_-\theta^0\theta^1 \ , \tag{6.305}$$

where the coefficients $\mathsf{A}_\pm$, B_0, and B_1 are functions of (w, z) only. From $\mathcal{S}\Psi = \overline{\mathcal{S}}\Psi = 0$, it follows that

$$\begin{aligned}
\left(\frac{\partial}{\partial w} + \frac{\partial \mathsf{W}}{\partial w}\right)\mathsf{A}_+ &= 0 \ , \\
\left(\frac{\partial}{\partial z} + \frac{\partial \mathsf{W}}{\partial z}\right)\mathsf{A}_+ &= 0 \ , \\
\left(\frac{\partial}{\partial w} + \frac{\partial \mathsf{W}}{\partial w}\right)\mathsf{B}_1 - \left(\frac{\partial}{\partial z} + \frac{\partial \mathsf{W}}{\partial z}\right)\mathsf{B}_0 &= 0 \ , \\
\left(\frac{\partial}{\partial w} - \frac{\partial \mathsf{W}}{\partial w}\right)\mathsf{B}_0 - \left(\frac{\partial}{\partial z} - \frac{\partial \mathsf{W}}{\partial z}\right)\mathsf{B}_1 &= 0 \ , \\
\left(\frac{\partial}{\partial w} - \frac{\partial \mathsf{W}}{\partial w}\right)\mathsf{A}_- &= 0, \\
\left(\frac{\partial}{\partial z} - \frac{\partial \mathsf{W}}{\partial z}\right)\mathsf{A}_- &= 0 \ ,
\end{aligned} \tag{6.306}$$

where W is given by (6.302).

These equations can be solved exactly [5] and the supersymmetric wave function has the form

$$\begin{aligned}
\Psi = \mathrm{e}^{-\mathsf{W}} + 2n\Upsilon^{1/4}\Big[& H_{n-1}\left(\Upsilon^{1/4}w\right) H_n\left(\Upsilon^{1/4}z\right)\theta^0 \\
& + H_n\left(\Upsilon^{1/4}w\right) H_{n-1}\left(\Upsilon^{1/4}z\right)\theta^1\Big]\mathrm{e}^{-\mathsf{W}} + \mathrm{e}^{\mathsf{W}}\theta^0\theta^1 \ ,
\end{aligned} \tag{6.307}$$

where $n \geq 0$. The solutions $\mathsf{A}_\pm$ represent the empty and filled fermion sectors of the wave function, respectively. Moreover, the empty fermion sector of this wave function is identical to the ground states $\Psi_{n=0}$ and $\Psi_{b=1}$ of the purely bosonic spectra discussed above. Hence, this supersymmetric approach to quantum cosmology naturally selects the bosonic state of lowest energy.

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[19] Due to the quantum Hamiltonian having an additional spin term, to avoid imaginary solutions to (6.301), we restrict the analysis to the region of parameter space where $\Upsilon > 0$. This corresponds to $\omega < -D/(D-1)$ when $\Lambda > 0$ and $\omega > -D/(D-1)$ if $\Lambda < 0$.

LRS Bianchi and Kantowski–Sachs Cosmologies

The above analysis can be extended [7] to anisotropic models. The metric for the class of spatially homogeneous LRS cosmological models with constant time hypersurfaces containing two-dimensional surfaces of constant curvature k is given by

$$ds^2 = -\mathcal{N}^2 dt^2 + e^{2\alpha-4\beta} dr^2 + e^{2\alpha+2\beta} d\Omega_{2,k}^2 \tag{6.308}$$

$$= -\mathcal{N}^2 dt^2 + a_1^2 dr^2 + a_2^2 d\Omega_{2,k}^2 \, , \tag{6.309}$$

where $\mathcal{N}$ is the lapse function, $d\Omega_{2,k}^2$ is the unit metric on the constant curvature 2D surfaces, $e^{3\alpha(t)} \equiv a_1 a_2^2$ determines the effective spatial volume of the universe and $\beta \equiv (1/3)\left[\ln(a_2/a_1)\right]$ determines the anisotropy of the model. The cases $k = \{-1, 0, +1\}$ correspond to the Bianchi type III, Bianchi type I, and Kantowski–Sachs universes, respectively.[20] Integrating over the spatial variables yields the minisuperspace action:

$$S = \int dt \mathcal{N} e^{3\alpha-\phi} \left(-6\frac{\dot{\alpha}^2}{\mathcal{N}^2} + 6\frac{\dot{\alpha}\dot{\phi}}{\mathcal{N}^2} + \omega\frac{\dot{\phi}^2}{\mathcal{N}^2} + \frac{\dot{\beta}^2}{\mathcal{N}^2} + 2k e^{-2\alpha-2\beta} - 2\Lambda \right) . \tag{6.310}$$

Introducing the *new* variables (throughout this section $\omega > -4/3$)

$$\varsigma \equiv \sqrt{\frac{3+2\omega}{4+3\omega}}(\phi - 3\alpha) \, , \tag{6.311}$$

$$u \equiv \sqrt{\frac{8+6\omega}{2+\omega}} \left\{ \frac{1}{4+3\omega}\left[\alpha + (1+\omega)\phi\right] + \beta \right\} \tag{6.312}$$

$$v \equiv \frac{1}{\sqrt{2+\omega}}\left[\alpha + (1+\omega)\phi - 2\beta\right] , \tag{6.313}$$

we find that we may diagonalise the kinetic sector of the reduced action (6.310):

$$S = \int dt \Bigg[\tfrac{1}{\mathcal{N}} e^{-\kappa_\omega \varsigma} \dot{u}^2 + \tfrac{1}{\mathcal{N}} e^{-\kappa_\omega \varsigma} \dot{v}^2 - \tfrac{1}{\mathcal{N}} e^{-\kappa_\omega \varsigma} \dot{\varsigma}^2 + 2\mathcal{N} k e^{(C-\kappa_\omega)\varsigma - \breve{G}u} - 2\mathcal{N}\Lambda e^{-\kappa_\omega \varsigma} \Bigg] , \tag{6.314}$$

where

$$\kappa_\omega \equiv \sqrt{\frac{4+3\omega}{3+2\omega}} \, , \tag{6.315}$$

[20] The geometry of the spatial sections of the Kantowski–Sachs model is $S^1 \times S^2$. The symmetry group of these surfaces is of the Bianchi type IX, but only acts transitively on 2D surfaces that foliate the three-space.

$$C \equiv \frac{2(1+\omega)}{\sqrt{(3+2\omega)(4+3\omega)}} \,, \tag{6.316}$$

$$\breve{G} \equiv \sqrt{\frac{4+2\omega}{4+3\omega}} \,. \tag{6.317}$$

Global symmetries in these models, corresponding to a generalization of scale factor duality, were first discussed in [59, 60]. The action (6.314) is invariant under the discrete Z_2 'duality' symmetry

$$\overline{u} = u \,, \qquad \overline{v} = -v \,, \qquad \overline{\varsigma} = \varsigma \,. \tag{6.318}$$

The classical Hamiltonian constraint is

$$\mathcal{H}_{\mathrm{c}} = -\pi_u^2 - \pi_v^2 + \pi_\varsigma^2 + 8k\mathrm{e}^{(C-2\kappa_\omega)\varsigma}\mathrm{e}^{-\breve{G}u} - 8\Lambda\mathrm{e}^{-2\kappa_\omega\varsigma} \,, \tag{6.319}$$

which can be written in the more compact form

$$\mathcal{H} = \mathcal{G}^{XY}\pi_X\pi_Y + U(q^X) \,, \tag{6.320}$$

$$U(q^X) = -8k\mathrm{e}^{(C-2\kappa_\omega)\varsigma}\mathrm{e}^{-\breve{G}u} + 8\Lambda\mathrm{e}^{-2\kappa_\omega\varsigma} \,, \tag{6.321}$$

where $q^X = (\varsigma, u, v)$ $(X = \{0, 1, 2\})$ and $\mathcal{G}^{XY} = \mathrm{diag}(-1, 1, 1)$ is the minisuperspace metric. By identifying the conjugate momenta with the operators $\pi_{q^X} = \pi_X = -\mathrm{i}\partial/\partial q^X$ and neglecting ambiguities that arise in the factor ordering, we arrive at the Wheeler–DeWitt equation:

$$\left[-\frac{\partial^2}{\partial\varsigma^2} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 8k\mathrm{e}^{(C-2\kappa_\omega)\varsigma}\mathrm{e}^{-\breve{G}u} - 8\Lambda\mathrm{e}^{-2\kappa_\omega\varsigma}\right]\Psi = 0 \,. \tag{6.322}$$

A supersymmetric extension requires a solution $\mathsf{W} = \mathsf{W}(q^X)$ of the Euclidean Hamilton–Jacobi equation:

$$\mathcal{G}^{XY}\frac{\partial\mathsf{W}}{\partial q^X}\frac{\partial\mathsf{W}}{\partial q^Y} = U(q^X) \,, \tag{6.323}$$

i.e.,

$$-\left(\frac{\partial\mathsf{W}}{\partial\varsigma}\right)^2 + \left(\frac{\partial\mathsf{W}}{\partial u}\right)^2 + \left(\frac{\partial\mathsf{W}}{\partial v}\right)^2 = -8k\mathrm{e}^{(C-2\kappa_\omega)\varsigma}\mathrm{e}^{-Gu} + 8\Lambda\mathrm{e}^{-2\kappa_\omega\varsigma} \,. \tag{6.324}$$

Proceeding similarly to (6.303), (6.304), for the 3D minisuperspace that we are considering, whose supersymmetric wave function can be expanded in terms of the Grassmann variables θ^a,

$$\Psi = \mathsf{A}_+ + \mathsf{B}_X\theta^X + \frac{1}{2}\varepsilon_{XYZ}\mathsf{C}^Z\theta^X\theta^Y + \mathsf{A}_-\theta^0\theta^1\theta^2 \,, \tag{6.325}$$

where the bosonic variables A_+, B_X, C_X, A_- are functions of the minisuperspace variables, ε_{XYZ} is totally antisymmetric on all its indices and $\varepsilon_{012} \equiv +1$, the Euclidean Hamilton–Jacobi equation for the LRS Bianchi type I cosmology is

$$-\left(\frac{\partial \mathsf{W}}{\partial \varsigma}\right)^2 + \left(\frac{\partial \mathsf{W}}{\partial u}\right)^2 + \left(\frac{\partial \mathsf{W}}{\partial v}\right)^2 = 8\Lambda \mathrm{e}^{-2\kappa_\omega \varsigma} . \tag{6.326}$$

A solution to (6.326) that respects the global symmetry and discrete Z_2 symmetry (6.318) of the reduced action (6.314) is given by

$$\Lambda < 0 : \qquad \mathsf{W} = \mp \frac{1}{\kappa_\omega}\left[\sqrt{A^2 - 8\Lambda x^2} - A \operatorname{cotanh}^{-1}\left(\frac{\sqrt{A^2 - 8\Lambda x^2}}{A}\right)\right] + A\sqrt{u^2 + v^2} , \tag{6.327}$$

$$\Lambda > 0 : \qquad \mathsf{W} = \mp \frac{1}{\kappa_\omega}\left[\sqrt{A^2 - 8\Lambda x^2} - A \tanh^{-1}\left(\frac{\sqrt{A^2 - 8\Lambda x^2}}{A}\right)\right] + A\sqrt{u^2 + v^2} , \tag{6.328}$$

where A is an arbitrary constant and $x \equiv \mathrm{e}^{-\kappa_\omega \varsigma}$. In the $\Lambda > 0$ case, one also requires $\varsigma \leq \ln(2\sqrt{2}\Lambda^{1/2}/A)$ if $A > 0$ and $\varsigma \geq \ln(-2\sqrt{2}\Lambda^{1/2}/A)$ for $A < 0$.

Given the solution (6.327), we could in principle quantize the system in a manifestly supersymmetric fashion.[21] The solution (6.327) then simplifies to

$$\mathsf{W} = \mp \frac{2\sqrt{-2\Lambda}}{\kappa_\omega} \mathrm{e}^{-\kappa_\omega \varsigma} , \tag{6.329}$$

and it follows immediately from (6.329) that we require $\Lambda \equiv -\Upsilon < 0$ for consistency. The SUSY constraints are then given by

$$\mathcal{S} = \mathrm{i}\frac{\partial}{\partial \theta^0}\frac{\partial}{\partial \varsigma} - \mathrm{i}\frac{\partial}{\partial \theta^1}\frac{\partial}{\partial u} - \mathrm{i}\frac{\partial}{\partial \theta^2}\frac{\partial}{\partial v} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\frac{\partial}{\partial \theta^0} \tag{6.330}$$

and

$$\overline{\mathcal{S}} = -\mathrm{i}\theta^0\frac{\partial}{\partial \varsigma} - \mathrm{i}\theta^1\frac{\partial}{\partial u} - \mathrm{i}\theta^2\frac{\partial}{\partial v} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\theta^0 , \tag{6.331}$$

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[21] For simplicity, however, we consider the case where $A = 0$. This corresponds to the limit $\varsigma \to -\infty$ and denotes a *weak coupling regime* for ς. In the strong limit ($\varsigma \to +\infty$), the term in A dominates and Λ can be positive. In the former situation, the 'averaged' scale factor volume (represented by α) and the dilaton are more important, while in the latter a large anisotropy dominates in the spatial directions. This allows Λ to be positive. In particular, one may have $\varsigma \leftarrow +\infty$ and $u \to -\infty$, with, e.g., the singularity $a_2 \to 0$, $a_1 \to \infty$.

respectively. It follows from (6.325) (see Exercise 6.8) that

$$\mathsf{A}_+ = \mathsf{A}_+^0 \mathrm{e}^{\mathsf{f}} \,, \tag{6.332}$$

where A_+^0 is an arbitrary constant and (following the notation in [7])

$$(\mathsf{W} =)\mathsf{f} \equiv \pm \frac{2\sqrt{2\lambda}}{\kappa_\omega} \mathrm{e}^{-\kappa_\omega \varsigma} \,. \tag{6.333}$$

Similarly,

$$\mathsf{A}_- = \mathsf{A}_-^0 \mathrm{e}^{-\mathsf{f}} \,, \tag{6.334}$$

where A_-^0 is a second arbitrary constant.

Redefining the functions B_X as

$$\mathsf{B}_X \equiv \breve{B}_X \mathrm{e}^{\mathsf{f}} \,, \qquad X = 0, 1, 2 \,, \tag{6.335}$$

and if $\breve{\mathsf{B}}_{1,2}$ are independent of ς, then these variables satisfy the 2D Laplace equation, subject to an integrability condition. $\breve{\mathsf{B}}_0$ is independent of $\{u, v\}$ and consistency leads to $\mathsf{B}_0 = \mathrm{e}^{-\mathsf{f}}$. If $\breve{\mathsf{C}}^{1,2}$ are independent of ς and satisfy the 2D Laplace equation and the integrability condition $\partial \breve{\mathsf{C}}^1/\partial u = -\partial \breve{\mathsf{C}}^2/\partial v$, then the function C^0 is given by the wave function for the empty fermion sector $\mathsf{C}^0 = \mathrm{e}^{\mathsf{f}}$.

Similarly to the situation in Sect. 6.2.3, when the general solution to the bosonic Wheeler–DeWitt equation depends only on the variable ς,

$$\Psi = c_1 I_0(\mathsf{f}) + c_2 K_0(\mathsf{f}) \,, \tag{6.336}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind with order zero, f is defined in (6.333), and c_i are arbitrary constants, it transpires that, in the large argument limit, the modified Bessel function of the first kind asymptotes to the form $I_0 \propto \mathsf{f}^{-1/2} \exp(\mathsf{f})$ and, consequently, there is a correlation, up to a negligible prefactor, with the fully bosonic component (6.332) of the supersymmetric wave function. Indeed, the solution $\mathsf{A}_+ = \exp(\mathsf{f})$ is an *exact* solution to the bosonic Wheeler–DeWitt equation if a suitable choice of factor ordering is made when identifying the momentum operator conjugate to the variable ς.

We now consider the supersymmetric quantization of the vacuum Kantowski–Sachs Brans–Dicke cosmology where $\Lambda = 0$ (see Exercise 6.9). The Wheeler–DeWitt and Euclidean Hamilton–Jacobi equations are given by

$$\left(-\frac{\partial^2}{\partial \varsigma^2} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 8\mathrm{e}^{A\varsigma + Bu} \right) \Psi = 0 \tag{6.337}$$

and

$$\left(\frac{\partial \mathsf{W}}{\partial \varsigma}\right)^2 - \left(\frac{\partial \mathsf{W}}{\partial u}\right)^2 - \left(\frac{\partial \mathsf{W}}{\partial v}\right)^2 = 8\mathrm{e}^{A\varsigma + Bu} , \tag{6.338}$$

respectively, where $A \equiv C - 2\kappa_\omega$ and $B \equiv -\breve{G}$.

Let us assume (because it enables us to derive the first and second order fermion states analytically) that the wave function does not depend on the variable v, and introduce 'null' variables over the reduced $(1+1)$-dimensional minisuperspace:

$$s \equiv \frac{8}{A^2 - B^2} \exp\left[\frac{1}{2}(A+B)(\varsigma + u)\right] ,$$
$$\tau \equiv \exp\left[\frac{1}{2}(A-B)(\varsigma - u)\right] . \tag{6.339}$$

The Euclidean Hamilton–Jacobi equation (6.338) becomes

$$\frac{\partial \mathsf{W}}{\partial s}\frac{\partial \mathsf{W}}{\partial \tau} = 1 , \tag{6.340}$$

and admits the solutions

$$\mathsf{W} = -bs - \frac{1}{b}\tau , \tag{6.341}$$

where $b = \mathrm{i}\mu$. Equation (6.341) is invariant under the duality transformation (6.318). Moreover, we see that the exact solution to the Wheeler–DeWitt equation (6.337) is also a WKB solution, $\Psi = \exp(\pm \mathsf{W})$, to the Euclidean Hamilton–Jacobi equation (6.338).

In performing the supersymmetric quantization of this cosmology, it is convenient to diagonalise[22] the minisuperspace metric. The reason is that the Grassmannian variables should satisfy the anticommuting relations $\{\psi^X, \overline{\psi}^Y\} = \mathcal{G}^{XY}$. We therefore introduce the pair of variables

$$T \equiv \frac{1}{2}(s+\tau) , \qquad X \equiv \frac{1}{2}(s-\tau) , \tag{6.342}$$

and this implies that the minisuperspace metric, defined in (6.320), has the non-trivial components $\mathcal{G}^{00} = -\mathcal{G}^{11} = -(A^2 - B^2)(T^2 - X^2)/2$ and $\mathcal{G}^{22} = 1$.

The supersymmetry constraints are then given by [7]

[22] A non-diagonal minisuperspace metric would mean that fermionic states could not be clearly separated after the wave function has been annihilated by the supercharges.

$$\mathcal{S} = -\mathrm{i}\mathcal{G}^{00}\frac{\partial}{\partial\theta^0}\frac{\partial}{\partial T} - \mathrm{i}\mathcal{G}^{11}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial X} - \mathrm{i}\mathcal{G}^{22}\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial v} + \mathrm{i}\mathcal{G}^{00}\frac{\partial}{\partial\theta^0}\frac{\partial \mathsf{W}}{\partial T} + \mathrm{i}\mathcal{G}^{11}\frac{\partial}{\partial\theta^1}\frac{\partial \mathsf{W}}{\partial X} \quad (6.343)$$

and

$$\overline{\mathcal{S}} = -\mathrm{i}\theta^0\frac{\partial}{\partial T} - \mathrm{i}\theta^1\frac{\partial}{\partial X} - \mathrm{i}\theta^2\frac{\partial}{\partial v} - \mathrm{i}\theta^0\frac{\partial \mathsf{W}}{\partial T} - \mathrm{i}\theta^1\frac{\partial \mathsf{W}}{\partial X}\ , \quad (6.344)$$

respectively, where

$$\mathsf{W} = -\left(b + \frac{1}{b}\right)T - \left(b - \frac{1}{b}\right)X\ . \quad (6.345)$$

For the supersymmetric wave function, we consider the new ansatz

$$\Psi = \mathsf{A}_+ + \mathsf{B}_X\theta^X + \frac{1}{2}\varepsilon_{XYZ}\mathsf{C}^Z\theta^X\theta^Y + \mathsf{A}_-\theta^0\theta^1\theta^2\ , \quad (6.346)$$

where $\mathsf{A}_\pm$, B_X, and C^Y are bosonic functions of the minisuperspace variables. The annihilation of the wave function (6.346) by the supercharges (SUSY generators) (6.343) and (6.344) yields a set of coupled, first-order partial differential equations.

The wave functions for the empty and filled fermion sectors are readily deduced:

$$\mathsf{A}_\pm = \mathrm{e}^{\mp\mathsf{W}}\ , \quad (6.347)$$

where W is given by (6.345). To solve the remaining equations, we assume that the amplitudes $\{\mathsf{B}_X, \mathsf{C}^Y\}$ are independent of the variable v. In this case, the general solution is $\mathsf{B}_2 = \exp(-\mathsf{W})$, modulo a constant of proportionality, where W is given by the Euclidean action (6.341). Likewise, $\gamma^2 = \exp(\mathsf{W})$.

The wave functions for the one-fermion sector are completely determined by transforming back to the null coordinate pair (s, τ) defined in (6.339). In terms of these variables, defining $\mathsf{Y} \equiv \mathsf{B}_0 - \mathsf{B}_1$ and $\mathsf{Z} \equiv b^{-1}(\mathsf{B}_0 + \mathsf{B}_1)$, one class of allowed solution is given by $\mathsf{Y} = \exp(-\mathsf{W})$ and $\mathsf{Z} = -b\mathsf{Y}$, where W is given by (6.341). The amplitudes $\mathsf{C}^{0,1}$ satisfy the unit-mass Klein–Gordon equation. We find that one class of consistent solutions is given by $\mathsf{R} = \exp(\mathsf{W})$ and $\mathsf{Q} = -b\mathsf{R}$.

To summarize, therefore, the supersymmetric wave function that we have found for the vacuum Brans–Dicke Kantowski–Sachs cosmology is given by [7]

$$\Psi = \mathrm{e}^{-\mathsf{W}} + \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1 + \mathrm{e}^{-\mathsf{W}}\theta^2 + \mathsf{C}^0\theta^1\theta^2 - \mathsf{C}^1\theta^0\theta^2 + \mathrm{e}^{\mathsf{W}}\theta^0\theta^1 + \mathrm{e}^{\mathsf{W}}\theta^0\theta^1\theta^2\ , \quad (6.348)$$

where $\mathsf{B}_{0,1}$ and $\mathsf{C}^{0,1}$ satisfy the unit-mass Klein–Gordon equation.

Summary and Review To end this chapter, we provide some revision points, which may be used to identify some of the main features and results of SQC with 'hidden' $N = 2$ SUSY:

1. How can 'hidden' $N = 2$ SUSY be retrieved (Sect. 6.1)?
2. How do the minisuperspaces here compare with those in Chap. 5, e.g., for the empty matter sector and then the cosmological constant case (Sects. 6.1.2 and 6.1.3)?
3. What about when (super)matter includes a scalar field or a Yang–Mills sector (Sects. 6.1.4 and 6.1.5)?
4. What new features does string theory bring into the context of SQC, in particular, in the context of this chapter (Sects. 6.2, 6.2.1, and 6.2.3)?

Problems

6.1 Conformal Factor Independence
Show that the superpotential $\mathsf{W}(q)$ is independent of the conformal factor $\tilde{\Omega}$.

6.2 Determining $\Psi_{N=2}$ for the k=1 FRW and Taub Models
Employing the tools in Sect. 6.1.2, find the quantum states that satisfy the equations $\mathcal{S}\Psi = 0$ and $\tilde{\mathcal{S}}\Psi = 0$ for the $k = 1$ FRW and Taub (microsuperspace sector) models.

6.3 Interpreting the FRW Solution
Discuss the FRW case using (6.31).

6.4 No General Conserved Probability Current
Verify that within $N = 2$ SQC, there is *no* general conserved probability current for Bianchi cosmologies, with Bianchi I being the trivial exception.

6.5 *N*=2 SUSY Bianchi Model with a Gauge Sector
Discuss the classical and quantum settings of an $N = 2$ SUSY Bianchi model with gauge vector fields [2].

6.6 *N*=2 Supersymmetry and Target Space Duality
Following [53], investigate how cosmological models retrieved from string theory (with target space duality) can acquire $N = 2$ SUSY.

6.7 *N*=2 Supersymmetry in the Dilaton–Gravity Scenario
For the case of FRW cosmology determine the empty and filled fermionic sectors for the wave function corresponding to hidden $N = 2$ supersymmetry.

6.8 *N*=2 Supersymmetric LRS Bianchi I Cosmology
Consider an LRS Bianchi I universe within a Brans–Dicke theory and find the wave function corresponding to hidden $N = 2$ supersymmetry.

6.9 N=2 Supersymmetric Kantowski–Sachs Cosmology

Discuss whether the Kantowski–Sachs wave function satisfies the Hawking–Page boundary conditions (relevant to a wormhole configuration) [28].

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Chapter 7
Obstacles and Results: From QC to SQC

Some of the research routes in supersymmetric quantum cosmology (SQC) are imported from the objectives and subsequent results within purely bosonic quantum cosmology (QC). More precisely, the overall aim for SQC has been (and still is) to employ a more fundamental framework, richer both in content as well as assumptions, where the structure of purely bosonic quantum cosmology fits consistently as a limiting situation. SQC will constitute a vaster, more adequate, and more elegant perspective for the very early universe. Let us point here to the 'essentials' in the QC programme, then to the relevant features of SUGRA. Afterwards we summarize the progress subsequently achieved in SQC.

7.1 Quantum Cosmology

Purely bosonic QC, which was briefly reviewed in Chap. 2 (for more thorough presentations, see [1–6]), has been faced with difficulties, and still is facing problems, which we discuss briefly below. Our knowledge of the very early universe will surely improve if we can resolve the following points:

1. How can we identify the initial cosmological state? For this, two fundamental laws are needed: (a) a basic dynamical law and (b) a law for a cosmological initial condition. But let us add a few comments about this statement. As described in Sect. 2.1, in a phase space description of cosmological dynamics, some features are strongly dependent on a specific choice of initial state. In addition, the subject of study is the whole universe, i.e., in cosmology the boundary conditions (necessary to solve the dynamical laws) for the evolution of the whole universe cannot be obtained from observations of a 'part' of the universe, 'outside' the ('sub')system under analysis, i.e., the universe itself. Hence, the cosmological boundary must be formulated within a fundamental law. The establishment of such a law is the guiding focus of quantum cosmology. And as the reader may be wondering right now, is there a fundamental principle that assists in determining such a law for the initial condition of the universe? In other words, *is there a selection rule for a boundary condition*?

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2. In the last point, to retrieve predictions from quantum cosmology, *a choice* of boundary conditions has to be made for the Wheeler–DeWitt equation, to select one of them from the class(es) of solutions. However, a few (severe?) obstacles emerge. To begin with, there are quite a few to choose from and it is not yet possible to distinguish between their observational differences [2, 7–10]. Another point is that, even in the case of the no-boundary proposal [11], it happens that a *unique* wave function is not selected. Indeed, there are many such wave functions, which *do not differ* in their semi-classical predictions.[1] To be more concrete, additional elements, surely of a more fundamental nature,[2] are required to yield a *unique* solution.
3. Assuming we achieve a satisfactory framework for both the dynamical and the initial conditions, what type of predictions could be made? Would they be testable, and if so how? In particular, can we establish that the most probable (classical) emergent spacetime will have a satisfactory inflationary phase, with suitable density fluctuations and gravitational waves?[3] At a more primordial level, can we then determine the probability distribution for the values of the constants of nature, dependent on the different choices of the vacuum state, possibly related to different compactifications[4] within string theory?
4. We should also consider the the well known *problem of time*. In canonical quantum gravity, a completely *static* description of the universe is provided within the wave function, containing all the information 'from beginning to end'. Time can only be introduced *intrinsically*, because the equation is hyperbolic (any configuration can constitute a time coordinate i.e., a clock field), hoping eventually to retrieve the emergence of the 'physical' time from a timeless equation.
5. There are also some mathematical weaknesses. The Wheeler–DeWitt equation for the full description of quantum gravity is *not* properly defined. The choice of Hilbert space is much less obvious and it is not clear whether it will become a well-defined operator-valued distribution (maybe loop quantum gravity [12–14] will provide some alternative insights).

Nevertheless, there is absolutely no reason for discouragement. On the contrary, in contrast to inaccurate rumors, quantum cosmology still offers a few, quite tenacious and deeply challenging problems for the years to come. The five points above are superb opportunities for research.

And SQC is indeed a vastly promising response to this call for exploration and explorers. SQC has been around for 30 years or so, having (implicitly) 'nucleated' in the early 1970s from the seminal work indicated in [16]. At the time it was not yet fully identifiable as a realization of a *supersymmetric* description

[1] Technically speaking, it all depends on the chosen complex contour where the integration is implemented.

[2] Could it be that SUSY would be such a principle?

[3] This means the *necessary* initial conditions for inflation must be *implied* by the boundary condition for the wave function of the universe.

[4] See the landscape problem within (quantum) string cosmology [15].

for the quantum mechanical universe, but the fascination and potential developments were already present at that genesis moment. In fact, soon after, significant results came to light, with alternative schools of thought involved in creative competition.

7.2 Supergravity

The strength of SQC arises from being based on the fundamental elements and results of supergravity (SUGRA) and superstring/M-theories, in the context of the canonical quantization. A significant point from the reduction of 4D $N = 1$ SUGRA to 1D models, through suitable homogeneous ansätze, is that it leads to cosmological minisuperspaces with $N = 4$ *local* supersymmetry [17, 1] (see, for example, Chap. 5 and references therein). Nevertheless, we must be aware of the severe reduction of degrees of freedom that a homogeneity truncation implies. The validity of the minisuperspace approximation in supersymmetric models remains an open problem (see, e.g., [18–20]).

We recall that $N = 1$ SUGRA is a theory with gauge invariances [21–26], i.e., it is invariant under local coordinate, Lorentz, and supersymmetry transformations, with the gauge invariances translated into constraints for the physical variables (see Appendix B). Hence, following the Dirac procedure [27–33] (in the canonical representation), the Hamiltonian and the diffeomorphism constraint, $\mathcal{H}_\perp$ and $\mathcal{H}^i$, respectively, associated with general coordinate transformations, together with the supersymmetry constraints, $\mathcal{S}^A$, corresponding to supersymmetry transformations, as well as the Lorentz constraints, $\mathcal{J}^{AB}$, for the Lorentz transformations, had to be determined. All physical states must satisfy these constraints [17, 1].

Then adopting the Dirac quantization procedure, we have shown how these constraints become operators applied to physical (state) wave functionals Ψ, and are subsequently equated to zero. A fundamental feature of the canonical formulation is that, in finding a physical state, the algebra of constraints implies that Ψ will consequently obey the Hamiltonian and momentum constraints, provided that the Lorentz and supersymmetry constraints are satisfied in advance. For this reason, $N = 1$ SUGRA is said to constitute a Dirac-like square root of the gravity constraints. In more technical terms, the analysis of the *second order* Wheeler–DeWitt equation (i.e., of the Klein–Gordon type) can be replaced by that of a Lorentz and supersymmetrically induced (!) set of *coupled first order* differential equations, by analogy with the well known Dirac equation. This could then have profound consequences [34] regarding the dynamics of the wave function of the universe.

However, the analysis (in particular, obtaining quantum physical solutions) of the general theory of SUGRA is a laborious assignment: one has *infinite* degrees of freedom [35, 36]. Hence, a sensible option has been to consider instead simple truncated (minisuperspace) models. In other words, SQC.

7.3 Supersymmetric Quantum Cosmology

The vast sector of the SQC research programme thus imports some of its guidelines from earlier investigations in quantum supergravity (see, e.g., [1, 37, 27, 30]). It has been gradually enlarged, with many cosmological scenarios extensively reported in the published literature [1, 38]. The main feature of SQC is to subscribe to the idea that treating both quantum gravity and supersymmetry effects as dominant will bring forward an improved description of the very early universe. This contrasts with 'conventional' (purely bosonic) quantum cosmology, where quantum gravity principles are present, but *not* supersymmetry. In the SQC framework, we will therefore find a larger set of variables (bosonic and fermionic), as well as additional symmetries which increase the number of constraints, subsequently imposing a wider algebra.

And so, has SQC provided any insight into any of the challenges and issues indicated in the previous sections? The answer is affirmative. We certainly have found interesting sets of solutions, some corresponding to what is present in quantum cosmology, as desired and as expected for reasons of consistency. In a certain sense, this could mean that supersymmetry allows us to *restrict and select* from more fundamental grounds, among the spectrum of solutions, with obvious implications for the 'choice' of boundary conditions. There are several schools of thought in SQC, each following different, perhaps complementary, but certainly active lines of research, some in competition for common goals (even employing somewhat unrelated methods). In spite of the difficulties, either intrinsic to SQC or imported from quantum cosmology, a rather exciting and uncharted landscape is available for research into the very early universe, where new horizons can be established and then pushed forward, creatively and tenaciously defining more original borders for our perspective.

7.3.1 FRW Models

FRW models are the simplest to analyse [39–41]. In Chap. 5, in the context of $N = 1$ SUGRA, we described how a suitable ansatz could be established for the metric and gravitino variables, in order to fulfill the geometric assumptions regarding the background spacetime. From there, it was possible to obtain the particular allowed Lorentz invariant SUSY wave functions of the universe in the empty case, and also in the presence of a cosmological constant, in particular, retrieving the no-boundary state within a wider set with $N = 4$ SUSY.

The next step was to include (super)matter, which was implemented firstly by means of a scalar supermultiplet. If only a kinetic term was present, the Hartle–Hawking and Hawking–Page (wormhole) solutions were found. However, in the presence of a generic superpotential, we found that non-trivial solutions [42] could only be retrieved with the assistance of an expansion of this superpotential in powers of a parameter $\ell \ll 1$. In this case, the constraints could be written in a power series. At order ℓ^0, we obtained the general class of supersymmetric states. This class includes in particular solutions present in the current literature (i.e., Hartle–Hawking [11] or wormhole [43] states). Furthermore, we found *other types* of solutions that have not yet been identified or investigated. At order ℓ^1, only a partial set of solutions could be retrieved. However, it is here that new solutions are found, and the dependence on the superpotential is manifestly displayed in the solutions. It should be stressed that these two sets of states (at order ℓ^0 and ℓ^1) may be employed as asymptotic branches to analyse the behavior of solutions at higher orders. More precisely, they could in particular be used in a numerical analysis of the general equations.

In summary, *new* non-trivial solutions do exist in SQC for FRW models, with superpotentials for scalar supermultiplets that are not equivalent to an effective cosmological constant. Alternatively, we also described how vector (i.e., non-Abelian gauge) fields can be dealt with in SQC. By making suitable ansätze for the vector fields and fermionic superpartners, we found that the full SUSY wave function had a sector which matched the results found in 'conventional' quantum cosmology, namely the Hartle–Hawking state.

But SUSY FRW models were also investigated from other angles. In Chap. 6, the FRW models were brought instead straight from a bosonic sector (e.g., of superstring theory) and then constructed to possess $N = 2$ SUSY (which was described as being hidden [44–48]). The interesting feature was the indication of WKB solutions with a complex (oscillatory) component, different from what was found in Chap. 5.

An important feature throughout all the investigations leading to the above results is that the fermion number defined by the Rarita–Schwinger (gravitino) field can be used as a good quantum number under *some* conditions, e.g., *no* cosmological constant. In fact, each sector with a fixed fermion number may be treated separately, but it should be emphasized that different fermions may be involved, e.g., the degree of freedom of the gravitino ψ_A and the fermionic partner χ_B of the (complex) scalar field ϕ. This property plays an important role in the analysis of Bianchi models.

7.3.2 Spatial Anisotropy

Bianchi models enabled us to consider anisotropic gravitational degrees of freedom and thus *more* gravitino modes [49–54, 38].

In Chap. 5, particular emphasis was put on supersymmetric Bianchi minisuperspaces, obtained from a truncation of the full theory of supergravity. In spite of some limitations (see Chap. 7 of Vol. II), these models did provide useful guidelines concerning the general theory. In fact, the canonical quantization of supersymmetric Bianchi models was initially confronted with troublesome prospects. Few or no physical states seemed to be allowed [49, 50, 53, 55, 54, 48]. This apparently implied that such minisuperspaces were useless models as far as the general theory was concerned [35]. But a refreshing breakthrough was then proposed [56, 51, 52, 57], providing the *correct* spectrum of solutions. Hence, the subject gained new momentum (see also [58]).

And, of course, a significant set of results and features was retrieved using the approach of Chap. 6, assigning (hidden) $N = 2$ SUSY to several models, with particular interest in those extracted from (super)string-related theories. Table 7.1 summarizes the type of solutions found so far by the canonical quantization of $N = 1$ and $N = 2$ supergravities (see the framework of Chap. 7 of Vol. II).

Table 7.1 Solutions found so far by the canonical quantization of $N = 1$ and $N = 2$ supergravities

Supergravity theory	$k = +1$ FRW	Bianchi class A	Full theory
Pure $N = 1$	HH, WH	HH, WH	WH
$N = 1$ with Λ	HH	CS $\longrightarrow$ WH, HH	?
$N = 2$	–	CS	CS
$N = 1$ with scalar fields	Not quite HH or WH	Not WH or HH	?
$N = 1$ with vector fields	HH, WH	?	?
$N = 2$ with general matter	$\Psi = 0$	?	?

HH, Hartle–Hawking no-boundary; WH, Hawking–Page wormhole; CS, Chern–Simons; ?, not yet found.

Hence, as we hope the reader will agree, the finale of this appraisal is wholly satisfactory. Moreover, many more problems and opportunities remain to be addressed within SQC. But this will be brought out in Chap. 8.

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Chapter 8
Routes Beyond the Borders

Arriving at this page, the reader has hopefully reached the conclusion that the quantization of supersymmetric cosmological models within supergravity and superstring theories constitutes a fascinating topic.

In Chap. 7, we described the essential features of SQC, along with relevant methods and corresponding results, and in particular the way some difficulties which seemed to foretell a disconcerting future were suitably remedied. However, the canonical quantization of supergravity and its application to cosmology is by *no* means a closed book. There are still (many) other problems that have not yet been satisfactorily addressed or even considered. In fact, these may constitute additional tests that supersymmetric quantum gravity and cosmology has to confront. By presenting and sharing them in this final chapter, we aim to further motivate the interested and resilient explorer to follow in our footsteps and go beyond.

Let us therefore consider this point, as Vol. I comes to a close, as an *intermediate stage*, and postpone a series of further tempting challenges.[1] We present a list of possible new lines of investigation, though briefly described, where methods, results, consequences, and implications are as yet unforeseen. In more precise terms, the interested reader will find here a selection of subsequent research directions, some simple, others requiring more time and eventually becoming part of graduate or post-graduate projects. If inspired, do feel free to contact and share your ideas with the author:

- Why has a comprehensible identification of the Hartle–Hawking solution been so problematic for FRW models in $N = 1$ supergravity with scalar supermultiplets.
- Obtaining conserved currents in supersymmetric minisuperspaces from Ψ [1, 2]. It seems that this is not possible except in very simple cases.
- The validity of the minisuperspace approximation in locally supersymmetric models.
- Improve on supersymmetric FRW models with just gauge fields from suitable ansätze for the vector *and* fermionic fields [3–6].

[1] Where the author and collaborators have currently been involved in some cases.

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- Study the canonical quantization of Kantowski–Sachs cosmologies (and black holes) in $N = 2$ and $N = 4$ supergravities [5, 7].
- Perform the canonical quantization of FRW models in $N = 3$ (or higher!) supergravity.
- Describe the results and features concerning finite probabilities for photon–photon scattering in $N = 2$ supergravity, but now from a canonical quantization point of view.
- Study the canonical quantization of supergravity theories in $d > 4$ dimensions.
- Study FRW supersymmetric minisuperspaces within a superspace formalism. It would be particularly interesting to consider Bianchi models and FRW models with supermatter, i.e., to see if the problems afflicting the FRW and Bianchi models mentioned still remain.
- Investigate whether there is a (more fundamental) relation connecting duality, SUSY, and integrability within SQC.
- The background space for general relativity is a Riemannian spacetime, but for SUGRA, can we be as explicit and clear geometrically speaking?
- Provide a full physical interpretation of all the terms in the Hamiltonian and momentum constraints in SUGRA.
- Can SQC be described with the assistance of Killing spinors [8]?
- Improve on the investigation of anti-de Sitter and possibly de Sitter states in FRW and Bianchi SUSY minusuperspaces.
- Improve on the description of FRW SQC with scalar supermultiplets and superpotential.
- Extend the analysis of Bianchi models in SQC, in particular, when there is a cosmological constant and superpotential (in the case of a scalar supermultiplet).
- What can SQC or the canonical quantization of $N = 1$ SUGRA tell us about the chaos scenario for the early universe?
- How can 'hidden' $N = 2$ SUSY be extended into an $N = 4$ framework and thereby related to the contents of Chap. 5? How can a conformal transformation eliminate quadratic and quartic fermionic terms [9, 10]?
- Can we find complex solutions to the SQC Hamilton–Jacobi equation and still have SUSY [9–14]?
- How can we deal with SUSY breaking in SQC?
- Can a suitable SQC be retrieved from more complicated settings, such as 10D superstring theories (via a *realistic* dimensional reduction)?
- Analyse *non*-diagonal Bianchi models in SQC.

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Appendix A
List of Symbols, Notation, and Useful Expressions

In this appendix the reader will find a more detailed description of the conventions and notation used throughout this book, together with a brief description of what spinors are about, followed by a presentation of expressions that can be used to recover some of the formulas in specified chapters.

A.1 List of Symbols

$\boldsymbol{\kappa} \equiv \kappa_{ijk}$	Contorsion
$\boldsymbol{\xi} \equiv \xi_{ijk}$	Torsion
K	Kähler function
$\mathsf{K}^I_J \equiv \mathrm{g}^I_J \equiv \mathsf{G}^I_J$	Kähler metric
$\mathsf{W}(\phi)$	Superpotential
V	Vector supermultiplet
ϕ, Φ	(Chiral) Scalar supermultiplet
ϕ, φ	Scalar field
$V(\phi)$	Scalar potential
$\overline{\chi} \equiv \gamma^0 \chi^\dagger$	For Dirac 4-spinor representation
$\psi^\dagger$	Hermitian conjugate (complex conjugate and transposition)
ϕ^*	Complex conjugate
$[M]^{\mathrm{T}}$	Transpose
$\{\ ,\ \}$	Anticommutator
$[\ ,\ ]$	Commutator
θ	Grassmannian variable (spinor)
$\mathcal{J}_{ab}, \mathcal{J}_{AB}$	Lorentz generator (constraint)
$q_X, X = 1, 2, \ldots$	Minisuperspace coordinatization
$\pi^{\mu\alpha\beta}$	Spin energy–momentum

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$S^{\mu\alpha\beta}$	Spin angular momentum
$\mathcal{D}$	Measure in Feynman path integral
$[\ ,\ ]_{\mathrm{P}} \equiv [\ ,\ ]$	Poisson bracket
$[\ ,\ ]_{\mathrm{D}}$	Dirac bracket
F	Superfield
M_{P}	Planck mass
$V\left(\beta^{+}, \beta^{-}\right)$	Minisuperspace potential (Misner–Ryan parametrization)
Z^{IJ}	Central charges
$\mathrm{d}s$	Spacetime line element
$\mathrm{d}\mathsf{s}$	Minisuperspace line element
$\mathcal{F}$	Fermion number operator
e_{μ}	Coordinate basis
$\mathbf{e}_{a}$	Orthonormal basis
${}^{(3)}\mathsf{V}$	Volume of 3-space
$v_{\mu}^{(a)}$	Vector field
$f_{\mu\nu}^{(a)}$	Vector field strength
π^{ij}	Canonical momenta to h_{ij}
π^{ϕ}	Canonical momenta to ϕ
π^{0}	Canonical momenta to $\mathcal{N}$
π^{i}	Canonical momenta to $\mathcal{N}_i$
p_{a}^{μ}	Canonical momenta to e_{μ}^{a} (tetrad)
P_{k}^{ij}	Canonical momenta to ξ_{ij}^{k} (torsion)
$\mathcal{J}_j$	Lorentz rotation generator
$\mathcal{K}_i$	Lorentz boost generator
$\mathcal{P}_{\mu}$	Translation (Poincaré) generator
$\varepsilon_{\mu\nu\lambda\sigma}$	Four-dimensional permutation
ε_{ijk}	Three-dimensional permutation
$\mathcal{S}_A$	SUSY generator (constraint)
$\mathcal{H}_{\perp}$	Hamiltonian constraint
$\mathcal{H}_i$	Momentum constraints
$\mathcal{G}$	DeWitt metric
$\mathcal{G}^{\mathrm{s}}$	DeWitt supermetric
Ψ	Wave function (state) of the universe
Ω	ADM time
$\mathcal{R}$	Spinor-valued curvature

$\beta_{ij}(\beta_+, \beta_-)$	Anisotropy matrices (Misner–Ryan representation) in Bianchi cosmologies
p_{ij}	Canonical momentum to β^{ij}
H_{ADM}	ADM Hamiltonian
K_{ij}	Extrinsic curvature
${}^{(4)}R$	Four-dimensional spacetime curvature
${}^{(3)}R$	Three-dimensional spatial curvature
Λ	Cosmological constant
$\Gamma^{\mu}_{\nu\lambda}$	Christoffel connection coefficients
,	Usual derivative (vectors, tensors)
$\vert$ and ${}^{(3)}\overline{D}_i$	Four-dimensional spacetime covariant derivative with respect to the 3-metric h_{ij}, *no* spin connection and no torsion, with Christoffel term (vectors, tensors)
; and ${}^{(4)}\overline{D}_\mu$	Four-dimensional spacetime covariant derivative with respect to the 4-metric $g_{\mu\nu}$, *no* spin connection and no torsion, with Christoffel term (vectors, tensors)
$\Vert$	Covariant derivative with respect to the 3-metric, including spin connection (vectors, tensors)
≀	Covariant derivative with respect to the 4-metric $g_{\mu\nu}$, including spin connection (vectors, tensors)
Dot over symbol	Proper time derivative
$\mathrm{d}\Omega_3^2$	Line element of spatial sections
k	Spatial curvature index
$a(t) \equiv \mathrm{e}^{\alpha(t)}$	FRW scale factor
$\mathcal{N}$	Lapse function
$\mathcal{N}^i$	Shift vector
$\mu, \nu, \ldots$	World spacetime indices
$\mathsf{a}, \mathsf{b}, \ldots$	Condensed superindices (either bosonic or fermionic)
$a, b, \ldots$	Local (Lorentz) indices
$\hat{a}, \hat{b}, \ldots = 1, 2, 3, \ldots$	Local (Lorentz) spatial indices
i, j, k	Spatial indices
h_{ij}	Spatial 3-metric
$g_{\mu\nu}$	Spacetime metric
η^{ab}	Lorentz metric
$\omega^\mu \equiv \{\omega^0, \omega^i\}$	One-form basis
A, A'	Two-spinor component indices (Weyl representation)

$[a]$	Four-spinor component indices, e.g., Dirac representation
$\mathsf{I}, \mathsf{J}, \ldots$	Kähler indices
J, j	Representation label of SU(2) (spin state)
M, N	Elements of SL(2,$\mathbb{C}$)
$\mathcal{L}_b^a$	Lorentz (transformation) matrix representation
$p^i_{AA'}$	Momentum conjugate to $e_i^{AA'}$
$\mathcal{H}_{AA'}$	Hamiltonian and momentum constraints condensed (two-spinor components)
S	Lorentzian action
L	Lagrangian
I	Euclidean action
$\mathbf{S}$	Superspace
$\mathcal{M}$	Four-dimensional spacetime manifold
Σ_t	Three-dimensional spatial hypersurface
t	Coordinate time
τ	Proper time
$\mathsf{n} \equiv \{n^\mu\}$	Normal to spatial hypersurface
${}^{(4)}D_\nu \equiv D_\nu$	Four-dimensional spacetime covariant derivative with respect to the 4-metric $g_{\mu\nu}$, *with* spin connection *and* torsion, *no* Christoffel term (spinors [vectors, tensors])
$\wr$ and $\mathbf{D}_\nu$	Four-dimensional spacetime covariant derivative with respect to the 3-metric h_{ij}, *with* spin connection *and* torsion, *with* Christoffel term (spinors [vectors, tensors])
$\|$	Three-dimensional analogue of $\mathbf{D}_\nu$ (spinors [vectors, tensors])
${}^{(3)}D_j \equiv {}^{(3)}\nabla_j \equiv \nabla_j$	Three-dimensional analogue of D_ν (spinors [vectors, tensors])
${}^{(3s)}D_j \equiv {}^{(3s)}\nabla_j \equiv \widetilde{\nabla}_j$	Three-dimensional analogue of D_ν, *no* torsion (spinors [vectors, tensors])
$\mathcal{M}_{ab}, \mathcal{M}_{AB}$	Lagrange multipliers for the Lorentz constraints (Lorentz and 2-spinor indices)
$\omega^A_{B\mu}, \omega^a_{b\mu}, \omega^{AA'BB'}$	Spin connections
ψ^A_μ	Gravitino field
$n^\mu, n^{AA'}$	Normal to the hypersurfaces
ε^{AB}	Metric for spinor indices (Weyl representation)
$\sigma_a^{AA'}$	Pauli matrices with two-component spinor indices (Weyl representation)

σ	Infeld–van der Waerden symbols
$e^{a}{}_{\mu}$	Tetrad
$\hat{e}^{\hat{a}}{}_{i}$	Spatial tetrad ($\hat{a}, i = 1, 2, 3$)
$e^{AA'}{}_{\mu}$	Tetrad with two-component spinor indices
$\cong$	Equality provided constraints are satisfied
(a)	Gauge group index
$\widetilde{D}_{\mu}, \check{\mathrm{D}}_{\phi}, \hat{D}_{\mu}$	Gauge group covariant derivatives
$\mathcal{Q}_{(a)}$	Gauge group charge operator
ε	Infinitesimal spinor
D_{ϕ}	Kähler derivative
D	SUSY covariant derivative (SUSY superspace)
$\Gamma_{\mathsf{I}}^{\mathsf{J}K}$	Kähler connection
R	Kähler curvature
ψ, χ, λ	Spinors
$\overline{\psi}, \overline{\chi}, \overline{\lambda}$	Conjugate spinors, in the 2-spinor SL(2,$\mathbb{C}$) Weyl representation
U	Minisuperspace potential
$\mathsf{P}(\phi, \overline{\phi})$	SUGRA-induced potential for scalar fields
$\mathcal{T}^{(a)}$	Gauge group generators
Q	SUSY supercharge operators
$\hat{G}$	Gauge group
T	Energy–momentum tensor
$\mathbf{D}$	Minisuperspace dimension
$\mathbf{R}$	Minisuperspace curvature
$D, \hat{d}, \hat{D}, \tilde{D}, \check{D}, \ldots$	Dimension of space
σ	Axion field
$\mathbf{1}, \boldsymbol{\sigma}$	Pauli matrices
$\mathsf{A}, \mathsf{B}, \ldots$	Bosonic amplitudes of SUSY Ψ

A.2 Conventions and Notation

Throughout this book we employ $c = 1 = \hbar$ and $G = 1 = M_{\mathrm{P}}^{-2}$, with $\mathsf{k} \equiv 8\pi G$ *unless otherwise indicated*. In addition, we take:

- $\mu, \nu, \ldots$ as world spacetime indices with values 0, 1, 2, 3,
- $a, b, \ldots$ as local (Lorentz) indices with values 0, 1, 2, 3,
- i, j, k as spatial indices with values 1, 2, 3,

- A, A' as 2-spinor notation indices with values 0, 1 or 0′, 1′, respectively,
- $[a]$ as 4-spinor component indices, e.g., the Dirac representation, with values 1, 2, 3, 4.

In this book we have also chosen the signature of the 4-metric $g_{\mu\nu}$ (or η_{ab}) to be $(-,+,+,+)$. Therefore, the 3-metric h_{ij} on the spacelike hypersurfaces has the signature $(+,+,+)$, which gives a positive determinant. Thus the 3D totally antisymmetric tensor density can be defined by $\epsilon^{0123} \equiv \epsilon^{123} = \epsilon_{123} = +1$.

A.3 About Spinors

As the reader may already have noticed (or will notice, if he or she is venturing into this appendix prior to probing more deeply into some of the chapters) there are some idiosyncrasies in the mathematical structures of SUSY and SUGRA which are essentially related to the presence of fermions (and hence of the spinors which represent them). But why is this, and what are the main features the spinors determine? In fact, why are spinors needed to describe fermions? What are spinors and how do they come into the theory? This section is devoted (in part) to introducing this issue.

Before proceeding, for completeness let us just indicate that a tetrad formalism is indeed mandatory for introduing spinors.[1] The tetrad corresponds to a massless spin-2 particle, the graviton [1], i.e., corresponding to two degrees of freedom.[2]

Note A.1 Extending towards a simple SUSY framework, with only one generator (labeled $N = 1$ SUSY), the corresponding (super)multiplet will also contain a fermion with spin 3/2, the gravitino, as described in Chap. 3, where the reader will find elements of SUSY (*not* in this appendix).

A.3.1 Spinor Representations of the Lorentz Group

Local Poincaré invariance is the symmetry that gives rise to general relativity [2]. It contains Lorentz transformations plus translations and is in fact a semi-direct product of the Lorentz group and the group of translations in spacetime. The Lorentz

[1] Tensor representations of the general linear group 4×4 matrices GL(4) behave as tensors under the Lorentz *sub*group of transformations, but there are *no* such representations of GL(4) which behave as spinors under the Lorentz (sub)group (see the next section).

[2] In simple terms, the tetrad has 16 components, but with four equations of motion, plus four degrees of freedom to be removed due to general coordinate invariance and six due to local Lorentz invariance, this leaves $16 - (4 + 4 + 6) = 2$.

group has six generators: three rotations $\mathcal{J}_{\hat{a}}$ and three boosts $\mathcal{K}_{\hat{b}}$, $\hat{a}, \hat{b} = 1, 2, 3$ with commutation relations [3–5]:

$$[\mathcal{J}_{\hat{a}}, \mathcal{J}_{\hat{b}}] = \mathrm{i}\varepsilon_{\hat{a}\hat{b}\hat{c}}\mathcal{J}_{\hat{c}} \ , \quad [\mathcal{K}_{\hat{a}}, \mathcal{K}_{\hat{b}}] = -\mathrm{i}\varepsilon_{\hat{a}\hat{b}\hat{c}}\mathcal{J}_{\hat{c}} \ , \quad [\mathcal{J}_{\hat{a}}, \mathcal{K}_{\hat{b}}] = \mathrm{i}\varepsilon_{\hat{a}\hat{b}\hat{c}}\mathcal{K}_{\hat{c}} \ . \tag{A.1}$$

The generators of the translations are usually denoted $\mathcal{P}_\mu$, with[3]

$$\left[\mathcal{P}_\mu, \mathcal{P}_\nu\right] = 0 \ , \quad \left[\mathcal{J}_i, \mathcal{P}_j\right] = \mathrm{i}\varepsilon_{ijk}\mathcal{P}_k \ , \tag{A.2}$$

$$[\mathcal{J}_i, \mathcal{P}_0] = 0 \ , \quad \left[\mathcal{K}_i, \mathcal{P}_j\right] = -\mathrm{i}\mathcal{P}_0\delta_{ij} \ , \quad [\mathcal{K}_i, \mathcal{P}_0] = -\mathrm{i}\mathcal{P}_i \ . \tag{A.3}$$

Or alternatively, defining the Lorentz generators $\mathcal{L}_{\mu\nu} \equiv -\mathcal{L}_{\nu\mu}$ as $\mathcal{L}_{0i} \equiv \mathcal{K}_i$ and $\mathcal{L}_{ij} \equiv \varepsilon_{ijk}\mathcal{J}_k$, the full Poincaré algebra reads

$$\left[\mathcal{P}_\mu, \mathcal{P}_\nu\right] = 0 \ , \tag{A.4}$$

$$\left[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}\right] = -\mathrm{i}\eta_{\nu\rho}\mathcal{L}_{\mu\sigma} + \mathrm{i}\eta_{\mu\rho}\mathcal{L}_{\nu\sigma} + \mathrm{i}\eta_{\nu\sigma}\mathcal{L}_{\mu\rho} - \mathrm{i}\eta_{\mu\sigma}\mathcal{L}_{\nu\rho} \ , \tag{A.5}$$

$$\left[\mathcal{L}_{\mu\nu}, \mathcal{P}_\rho\right] = -\mathrm{i}\eta_{\rho\mu}\mathcal{P}_\nu + \mathrm{i}\eta_{\rho\nu}\mathcal{P}_\mu \ . \tag{A.6}$$

Let us focus on the mathematical structure of (A.1). The usual tensor (vector) formalism and corresponding representation is quite adequate to deal with most situations of relativistic classical physics, but there are significant advantages in considering a more general exploration, namely from the perspective of the theory of representations of the Lorentz group. The *spinor representation* is very relevant indeed. For this purpose, it is usual to discuss how, under a general (infinitesimal) Lorentz transformation, a *general* object transforms linearly, decomposing it into *irreducible* pieces. To do this, we take the linear combinations

$$\mathcal{J}_j^{\pm} \equiv \frac{1}{2}(\mathcal{J}_j \pm \mathrm{i}\mathcal{K}_j) \ , \tag{A.7}$$

and the Lorentz algebra separates into *two* commuting SU(2) algebras:

$$[\mathcal{J}_i^{\pm}, \mathcal{J}_j^{\pm}] = \mathrm{i}\varepsilon_{ijk}\mathcal{J}_k^{\pm} \ , \qquad [\mathcal{J}_i^{\pm}, \mathcal{J}_j^{\mp}] = 0 \ , \tag{A.8}$$

i.e., each corresponding to a full angular momentum algebra. A few comments are then in order:

- The representations of SU(2) are known, each labelled by an index j, with $\mathsf{j} = 0, 1/2, 1, 3/2, 2, \ldots$, giving the spin of the state (in units of $\hbar$). The spin-j representation has dimension $(2\mathsf{j} + 1)$.

---

[3] Note that we will be using either world (spacetime) indices μ, or local Lorentz indices $a, b, \ldots$, and also world (space) indices i, or local spatial Lorentz indices $\hat{a}, \hat{b}, \ldots$, which can be related through the tetrad e^a_μ (see Sect. A.2).

- The representations of the Lorentz algebra (A.8) can therefore be labelled by[4] $(\mathsf{j}_-, \mathsf{j}_+)$, the dimension being $(2\mathsf{j}_+ + 1)(2\mathsf{j}_- + 1)$, where we find states with j taking integer steps between $|\mathsf{j}_+ - \mathsf{j}_-|$ and $\mathsf{j}_+ + \mathsf{j}_-$.
  - In particular, $(0, 0)$ is the scalar representation, while $(0, 1/2)$ and $(1/2, 0)$ both have dimension two for spin $1/2$ and constitute *spinorial* representations.[5]
  - In the $(1/2, 0)$ representation, $\mathsf{J}^- \equiv \{\mathcal{J}_i^-\}_{i=1,\ldots}$ can be represented by 2×2 matrices and $\mathsf{J}^+ = 0$, viz., $\mathsf{J}^- = \boldsymbol{\sigma}\backslash 2$, which implies[6] $\mathsf{J} = \boldsymbol{\sigma}\backslash 2$, but $\mathsf{K} = \mathrm{i}\boldsymbol{\sigma}\backslash 2$, where we henceforth use[7] the four 2×2 matrices $\sigma_\mu \equiv (\mathbf{1}, \sigma^i)$, with σ_0 usually being the identity matrix and $\boldsymbol{\sigma} \equiv \sigma_i$, $i = 1, 2, 3$, the three Pauli matrices:

$$\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{(A.9)}$$

    satisfying $\left[\sigma^i, \sigma^j\right] = 2\mathrm{i}\varepsilon^{ijk}\sigma^k$. Note the relation between Pauli matrices with upper and lower indices, namely $\sigma^0 = -\sigma_0$ and $\sigma^i = \sigma_i$, using the metric $\eta_{\mu\nu}$.
  - In the $(0, 1/2)$ representation we have $\mathsf{J} = \boldsymbol{\sigma}\backslash 2$ but $\mathsf{K} = -\mathrm{i}\boldsymbol{\sigma}\backslash 2$.

  This means that we have two types of spinors, associated respectively with $(0, 1/2)$ and $(1/2, 0)$, which are *inequivalent* representations. These will correspond in the following to the *primed* spinor $\overline{\psi}_{A'}$ and the *unprimed* spinor ψ_A, where $A = 0, 1$ and $A' = 0', 1'$ [11–13, 7].

The ingredients above conspire to make the group[8] SL(2,ℂ) the universal cover[9] of the Lorentz group. If M is an element of SL(2,ℂ), so is $-\mathsf{M}$, and both produce the same Lorentz transformation.

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[4] The pairs $(\mathsf{j}_-, \mathsf{j}_+)$ of the finite dimensional *irreducible* representations can also be extracted from the eigenvalues $\mathsf{j}_\pm(\mathsf{j}_\pm + 1)$ of the two Casimir operators $\mathsf{J}_\pm^2$ (operators proportional to the identity, with the proportionality constant labelling the representation).

[5] The representation $(1/2, 1/2)$ has components with $\mathsf{j} = 1$ and $\mathsf{j} = 0$, i.e., the spatial part and time component of a 4-vector. Moreover, taking the tensor product $(1/2, 0) \otimes (0, 1/2)$, we can get a 4-vector representation.

[6] It should be noted from $\mathsf{J} = \boldsymbol{\sigma}\backslash 2$ that a spinor effectively rotates through half the angle that a vector rotates through (spinors are periodic only for 4π).

[7] The description that follows is somewhat closer to [6–8], but different authors use other choices (see, e.g., [3, 5, 9, 10]), in particular concerning the metric signature, σ_0, and some spinorial elements and expressions. See also Sect. A.4.

[8] The letter S stands for 'special', indicating unit determinant, and the L for 'linear', while ℂ denotes the complex number field, whence SL(2,ℂ) is the group of 2×2 complex-valued matrices.

[9] So the Lorentz group is SL(2,ℂ)/Z_2, where Z_2 consists of the elements 1 and -1. Note also that SU(2) is the universal covering for the spatial rotation group SO(3). There is a *two-to-one* mapping of SU(2) onto SO(3).

A spinor is therefore the object carrying the basic representation of SL(2,$\mathbb{C}$), fundamentally constituting a complex 2-component object (e.g., 2-component spinors, or Weyl spinors)

$$\psi_A \equiv \begin{bmatrix} \psi_{A=0} \\ \psi_{A=1} \end{bmatrix}$$

transforming under an element M according to

$$\psi_A \longrightarrow \psi'_A = \mathsf{M}^B{}_A \psi_B \ , \qquad \mathsf{M} \equiv \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C}) \ , \tag{A.10}$$

with $A, B = 0, 1$, labelling the components. The peculiar feature is that now the other 2-component object $\overline{\psi}_{A'}$ transforms as

$$\overline{\psi}_{A'} \longrightarrow \overline{\psi}'_{A'} = M^{*B'}{}_{A'} \overline{\psi}_{B'} \ , \tag{A.11}$$

which is the primed spinor introduced above, while the above ψ_A is the unprimed spinor.

Note A.2 The representation carried by the ψ_A is $(1/2, 0)$ (matrices M). Its complex conjugate $\overline{\psi}_{A'}$ in $(0, 1/2)$ is *not* equivalent: M and M^* constitute inequivalent representations, with the complex conjugate of (A.10) associated with (A.11), and $\overline{\psi}_{A'}$ *identified* with $(\psi_A)^*$.

Note A.3 There is *no* unitary matrix U such that $\mathsf{N} = \mathsf{U}\mathsf{M}\mathsf{U}^{-1}$ for matrices $\mathsf{N} \equiv \mathsf{M}^*$. We have instead $\mathsf{N} = \zeta \mathsf{M}^* \zeta^{-1}$ with

$$\zeta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \equiv -\mathrm{i}\sigma_2 \ .$$

This follows from the (formal) relation $\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}$, which suggests writing ζ in terms of a known matrix (appropriate for 2-component spinors), viz., σ_2.

It is from this feature applied to a Dirac 4-spinor

$$\psi \equiv \begin{pmatrix} \chi \\ \overline{\eta} \end{pmatrix} \ ,$$

that Lorentz invariants are obtained as $(\mathrm{i}\sigma^2\chi)^{\mathrm{T}}\chi$. (This is the Weyl representation, where $\chi, \overline{\eta}$ are 2-component spinors transforming with M and N, respectively.) To be more concrete, with

$$\chi \equiv \chi_A = \begin{pmatrix} \chi_{A=0} \\ \chi_{A=1} \end{pmatrix} ,$$

we put

$$\chi^A = \begin{pmatrix} \chi^0 \\ \chi^1 \end{pmatrix} = \mathrm{i}\sigma^2 \chi = \begin{pmatrix} \chi_1 \\ -\chi_0 \end{pmatrix} ,$$

whence the invariant is $\chi^A \chi_A$. (Note that $\chi^0\chi_0 + \chi^1\chi_1 = \chi_1\chi_0 - \chi_0\chi_1 = -\chi_1\chi^1 - \chi_0\chi^0 = -\chi_A\chi^A$, which is quite different from the situation for spacetime vectors and tensors $A_\mu A^\mu = A^\mu A_\mu$!) This can be simplified by introducing *new* matrices with $\chi^A = \varepsilon^{AB}\chi_B$, where (formally!) $\varepsilon^{AB} \equiv \mathrm{i}\sigma^2$. Likewise for

$$\overline{\eta} \equiv \overline{\eta}^{A'} = \begin{pmatrix} \overline{\eta}^{0'} \\ \overline{\eta}^{1'} \end{pmatrix} ,$$

$\overline{\eta}^{A'}\overline{\eta}_{A'}$, $\overline{\eta}^{A'} = \varepsilon^{A'B'}\overline{\eta}_{B'}$, $\varepsilon^{A'B'} \equiv \mathrm{i}\sigma^2$, etc. Of course, $\varepsilon^{AB} \equiv \mathrm{i}\sigma^2$, $\varepsilon^{A'B'} \equiv \mathrm{i}\sigma^2$, are *different* objects acting on *different* spinors and spaces. The presence of the matrix representation is formal for computations.

The primed and unprimed index structure[10] carries to the generators, e.g., $\mathsf{J} = \boldsymbol{\sigma}\backslash 2$ and $\mathsf{K} = \mathrm{i}\boldsymbol{\sigma}\backslash 2$, with the four σ_μ matrices

$$(\sigma^\mu)_{AA'} = \{-\mathbf{1}, \sigma^i\}_{AA'} , \tag{A.12}$$

and

$$(\overline{\sigma}^\mu)^{A'A} \equiv \varepsilon^{A'B'}\varepsilon^{AB}(\sigma^\mu)_{BB'} \equiv (\mathbf{1}, -\sigma^i)^{A'A} , \tag{A.13}$$

where indices are raised by the antisymmetric 2-index tensors ε^{AB} and ε_{AB} given by[11]

$$\varepsilon^{AB} = \varepsilon^{A'B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \qquad \varepsilon_{AB} = \varepsilon_{A'B'} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \tag{A.14}$$

with which:

$$\psi^A = \varepsilon^{AB}\psi_B , \quad \psi_A = \varepsilon_{AB}\psi^\beta , \qquad \overline{\psi}^{A'} = \varepsilon^{A'B'}\overline{\psi}_{B'} , \qquad \overline{\psi}_{A'} = \varepsilon_{A'B'}\overline{\psi}^{B'} . \tag{A.15}$$

[10] Note that (A.12) and (A.13) can used to convert an O(1,3) vector into an SL(2,$\mathbb{C}$) mixed bispinor $\mathsf{b}_{AB'}$.

[11] Note that, in a *formal* matrix form, $\varepsilon^{AB} = \varepsilon^{A'B'} = \mathrm{i}\sigma_2$, $\varepsilon_{AB} = \varepsilon_{A'B'} = -\mathrm{i}\sigma^2$.

Note A.4 Note the difference between $(\sigma^\mu)_{AA'}$ and $(\overline{\sigma}^\mu)^{A'A}$, i.e., the lower, upper and respective order of indices, arising from $\varepsilon^{AB} = \varepsilon^{A'B'} = \mathrm{i}\sigma_2$ and $\varepsilon_{AB} = \varepsilon_{A'B'} = -\mathrm{i}\sigma^2$. To be more precise, the ('natural') lower (upper) $A'A$ indices on $(\sigma^\mu)_{AA'}$, $(\overline{\sigma}^\mu)^{A'A}$ come from covariance, e.g., analysing how the matrices

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix}$$

transform under a Lorentz transformation (the Weyl representation here). Further relations [7], in particular between σ^μ and $\overline{\sigma}^\mu$, can be written

$$\sigma_{\mu CD'}\overline{\sigma}^{\mu A'B} = -2\delta_C^B\delta_{D'}^{A'} \,, \tag{A.16}$$

$$\mathrm{Tr}\left(\sigma^\mu\overline{\sigma}^\nu\right) \equiv \sigma^\mu_{AB'}\overline{\sigma}^{\nu B'A} = -2g^{\mu\nu} \,, \tag{A.17}$$

constituting two *completeness* relations.

In essence, it all involves the statement that σ^μ_{AB} is Hermitian, and that, for a *real* vector, its corresponding spinor is also Hermitian.

Note A.5 In addition, the following points may be of interest:

- Concerning (A.14) and (A.15), it must be said that this is a matter of convention [5, 12, 13, 10, 7].
- Unprimed indices are always contracted from upper left to lower right ('ten to four'), while primed indices are always contracted from lower left to upper right ('eight to two'). This rule does *not* apply when raising or lowering spinor indices with the ε tensor.
- However, other choices can be made (see Sect. A.4), and this is what we actually follow from Chap. 4 onwards. In particular,

$$\varepsilon^{AB} = \varepsilon^{A'B'} = \varepsilon_{AB} = \varepsilon_{A'B'} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \,, \tag{A.18}$$

with which (note the *position* of terms and *order* of indices, in contrast with (A.14)):

$$\psi^A = \varepsilon^{AB}\psi_B \,, \quad \psi_A = \psi^B\varepsilon_{BA} \,, \quad \overline{\psi}^{A'} = \varepsilon^{A'B'}\overline{\psi}_{B'} \,, \quad \overline{\psi}_{A'} = \overline{\psi}^{B'}\varepsilon_{B'A'} \,. \tag{A.19}$$

With (A.15) (or (A.19)), scalar products such as $\psi\chi$ or $\overline{\psi}\,\overline{\chi}$ can be employed. Of course, when more than one spinor is present, we have to remember that spinors anticommute.[12] Therefore, for 2-component spinors, $\psi_0\chi_1 = -\chi_1\psi_0$, and also, e.g., $\psi_0\overline{\chi}_{1'} = -\overline{\chi}_{1'}\psi_0$. In more detail [7],

$$\psi\chi \equiv \psi^A\chi_A = \varepsilon^{AB}\psi_B\chi_A = -\varepsilon^{AB}\psi_A\chi_B$$

$$= -\psi_A\chi^A = \chi^A\psi_A = \chi\psi \ , \tag{A.20}$$

$$\overline{\psi}\,\overline{\chi} \equiv \overline{\psi}_{A'}\overline{\chi}^{A'} = \ldots = \overline{\chi}_{A'}\overline{\psi}^{A'} = \overline{\chi}\,\overline{\psi} \ , \tag{A.21}$$

$$(\psi\chi)^\dagger \equiv (\psi^A\chi_A)^\dagger = \overline{\chi}_{A'}\overline{\psi}^{A'} = \overline{\chi}\,\overline{\psi} = \overline{\psi}\,\overline{\chi} \ , \tag{A.22}$$

as well as

$$\psi\sigma^\mu\overline{\chi} = \psi^A\sigma^\mu_{AB'}\overline{\chi}^{B'} \ , \qquad \overline{\psi}\,\overline{\sigma}^\mu\chi = \overline{\psi}_{A'}\overline{\sigma}^{\mu A'B}\chi_B \ , \tag{A.23}$$

from which, e.g.,

$$\chi\sigma^\mu\overline{\psi} = -\overline{\psi}\,\overline{\sigma}^\mu\chi \ , \tag{A.24}$$

$$\chi\sigma^\mu\overline{\sigma}^\nu\psi = \psi\sigma^\nu\overline{\sigma}^\mu\chi \ , \tag{A.25}$$

$$(\chi\sigma^\mu\overline{\psi})^\dagger = \psi\sigma^\mu\overline{\chi} \ , \tag{A.26}$$

$$(\chi\sigma^\mu\overline{\sigma}^\nu\psi)^\dagger = \overline{\psi}\sigma^\nu\overline{\sigma}^\mu\overline{\chi} \ , \tag{A.27}$$

A.3.2 Dirac and Majorana Spinors

The above (Weyl) framework will often be used for most of this book (in particular, (A.18) and (A.19)), but if the reader goes on to study the fundamental literature on SUSY and SUGRA, other representations will be met, some of which are also adopted by a few authors in SQC. Let us therefore comment on that (see Chap. 7 of Vol. II). In particular, we present the Dirac and Majorana spinors (associated with another representation).

A 4-component Dirac spinor is made from a 2-component unprimed spinor and a 2-component primed spinor via

$$\psi_{[a]} \equiv \begin{pmatrix} \psi_A \\ \overline{\chi}^{A'} \end{pmatrix} \ ,$$

---

[12] The components of two-spinors are *Grassmann numbers*, anticommuting among themselves. Therefore, complex conjugation includes the reversal of the order of the spinors, e.g., $(\zeta\psi)^* = (\zeta^A\psi_A)^* = (\psi_A)^*\left(\zeta^A\right)^* = \overline{\psi}_{A'}\overline{\zeta}^{A'} = \overline{\psi}\,\overline{\zeta}$.

transforming as the reducible $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, hence with

$$\begin{pmatrix} \psi_A \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \overline{\chi}^{A'} \end{pmatrix}$$

as Weyl spinors.[13]

But is there any need for a Dirac spinor (apart from in the Dirac equation)? The point is that, under a *parity* transformation [3, 5], the boost generators K change sign, whereas the generators J do not. It follows that the $(\mathsf{j}, 0)$ and $(0, \mathsf{j})$ representations are interchanged under parity, whence 2-component spinors are not sufficient to provide a full description. The Dirac 4-component spinor carries an irreducible representation of the Lorentz group *extended by parity*.

Note A.6 In addition, from

$$\psi_{[a]} \equiv \begin{pmatrix} \chi_A \\ \overline{\eta}^{A'} \end{pmatrix} \Longrightarrow \psi^{\dagger}_{[a]} = \left(\overline{\chi}_{A'}, \eta^A\right) , \tag{A.28}$$

the *adjoint* Dirac spinor is written as (Weyl representation)

$$\overline{\psi}_{[a]} \equiv -\psi^{\dagger}\gamma^0 \left(\eta^A, \overline{\chi}_{A'}\right) \Longrightarrow \overline{\psi}^{\mathrm{T}}_{[a]} = \begin{pmatrix} \eta^A \\ \overline{\chi}_{A'} \end{pmatrix} , \quad \gamma^0 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{A.29}$$

Note that $\overline{\psi}_1\psi_2 \equiv \eta_1\chi_2 + \overline{\chi}_1\overline{\eta}_2$ where the right-hand side is a Weyl 2-spinor representation, noticing that in the left-hand side we have Dirac 4-spinors, with the bar above either ψ_1 or χ_1 concerning different configurations and operations, and subscripts 1 and 2 merely labeling different Dirac spinors.

Note A.7 The *charge conjugate* spinor ψ^{c} is defined by

$$\psi^{\mathrm{c}} \equiv C\overline{\psi}^{\mathrm{T}} = \begin{pmatrix} -\mathrm{i}\sigma^2\eta^A \\ \mathrm{i}\sigma^2\overline{\chi}_{A'} \end{pmatrix} = \begin{pmatrix} \eta_A \\ \overline{\chi}^{A'} \end{pmatrix} , \tag{A.30}$$

where

$$C \equiv \begin{bmatrix} \varepsilon_{AB} = -\mathrm{i}\sigma^2 & 0 \\ 0 & \varepsilon^{A'B'} = \mathrm{i}\sigma^2 \end{bmatrix} . \tag{A.31}$$

---

[13] There are also *chiral* Dirac spinors. These constitute eigenstates of γ^5 and behave *differently* under Lorentz transformations (see (A.33) and (A.34)).

Note A.8 The *Majorana condition* $\psi = \psi^{\mathrm{c}}$ gives $\chi = \eta$:

$$\psi = \psi^{\mathrm{c}} = \begin{pmatrix} \chi_A \\ \overline{\chi}^{A'} \end{pmatrix} . \tag{A.32}$$

The importance of the Majorana condition is that it defines the *Majorana spinor*. This is invariant under charge conjugation, therefore constituting a relation (a sort of 'reality' condition) between ψ and the (complex) conjugate spinor $\psi^{\dagger}$. A Majorana spinor is a *particular* Dirac spinor, namely

$$\begin{pmatrix} \psi_A \\ \overline{\psi}^{A'} \end{pmatrix} ,$$

with only half as many independent components. A Majorana spinor is therefore a Dirac spinor for which the Majorana condition (A.32) is implemented: $\psi = \psi^{\mathrm{c}}$, i.e., $\overline{\chi}_{A'} = \psi^{\dagger}_{A}$.

Of course, whenever we have (Dirac or Majorana) 4-spinors, a related matrix framework must be used, in the form of the Dirac matrices (here in the chiral or Weyl representation):

$$\gamma^{\mu} \equiv \begin{bmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{bmatrix} , \qquad \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} , \qquad \left(\gamma^5\right)^2 = \mathbf{1} , \tag{A.33}$$

with γ_5 inducing the projection operators $(\mathbf{1} \pm \gamma_5)/2$ for the chiral components ψ_A, $\overline{\chi}^{A'}$ of ψ, determining their helicity as right (positive) or left (negative). But there are many *other* options [7]:

- Standard[14] canonical basis:

$$\gamma_0 \equiv \begin{bmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix} , \qquad \gamma_i \equiv \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} . \tag{A.34}$$

- A Majorana basis, where $\gamma_i^* = -\gamma_i$:

$$\begin{gathered} \gamma_0 \equiv \begin{bmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} , \qquad \gamma_1 \equiv \begin{bmatrix} 0 & \mathrm{i}\sigma^3 \\ \mathrm{i}\sigma^3 & 0 \end{bmatrix} , \\ \gamma_2 \equiv \begin{bmatrix} \mathrm{i}\mathbf{1} & 0 \\ 0 & -\mathrm{i}\mathbf{1} \end{bmatrix} , \qquad \gamma_3 \equiv \begin{bmatrix} 0 & -\mathrm{i}\sigma^1 \\ -\mathrm{i}\sigma^1 & 0 \end{bmatrix} . \end{gathered} \tag{A.35}$$

[14] The *standard* (Dirac) representation has γ_0 appropriate to describe particles (e.g., plane waves) in the rest frame.

- A basis in which the γ matrices are all real and where $N = 1$ SUGRA allows a real Rarita–Schwinger (gravitino) field (see Sect. 4.1) [14]:

$$\gamma_0 \equiv \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix} , \quad \gamma_1 \equiv \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} , \quad \gamma_2 \equiv \begin{bmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{bmatrix} ,$$
$$\gamma_3 \equiv \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} , \quad \gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & \mathrm{i}\sigma^3 \\ -\mathrm{i}\sigma^3 & 0 \end{bmatrix} . \tag{A.36}$$

Furthermore,

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} . \tag{A.37}$$

Through (A.33), the Lorentz generators then become

$$\mathcal{L}^{\mu\nu} \longrightarrow \frac{\mathrm{i}}{2}\gamma^{\mu\nu} ,$$

where

$$\gamma^{\mu\nu} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{bmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{bmatrix} , \tag{A.38}$$

and $\{,\}$ and $[,]$ denote the anticommutator and commutator, respectively, As expected from the representation $(1/2, 0) \oplus (0, 1/2)$, this determines that the ψ_A and $\overline{\chi}^{A'}$ spinors transform separately. The specific generators are also rewritten $\mathrm{i}\sigma^{\mu\nu}$ for ψ_A and $\mathrm{i}\overline{\sigma}^{\mu\nu}$ for $\overline{\chi}^{A'}$, with

$$(\sigma^{\mu\nu})_A{}^B \equiv \frac{1}{4}\left[\sigma^\mu_{AC'}\overline{\sigma}^{\nu C'B} - (\mu \leftrightarrow \nu)\right] , \tag{A.39}$$
$$(\overline{\sigma}^{\mu\nu})_{B'}{}^{A'} \equiv \frac{1}{4}\left[\overline{\sigma}^{\mu A'C}\sigma^\nu_{CB'} - (\mu \leftrightarrow \nu)\right] . \tag{A.40}$$

A.4 Useful Expressions

In this section we present a set of formulas which will help to clarify some properties of SUSY and SUGRA, and help also to simplify some of the expressions in the book. The initial emphasis is on the 2-component spinor, but we will also indicate, whenever relevant, some specific expressions involving 4-component spinors.

In a 2-component spinor notation, we can use a spinorial representation for the tetrad. This then allows us to deal with the indices of bosonic and fermionic variables in a suitable *equivalent* manner. To be more precise, in flat space we therefore associate a spinor with any vector by the Infeld–van der Waerden symbols $\sigma_a^{AA'}$, which are given by [15, 12, 13, 7][15]

$$\sigma_0^{AA'} = -\frac{1}{\sqrt{2}}\mathbf{1} , \qquad \sigma_i^{AA'} = \frac{1}{\sqrt{2}}\sigma_i \ . \tag{A.41}$$

Here, $\mathbf{1}$ denotes the unit matrix and σ_i are the three Pauli matrices (A.9). To raise and lower the spinor indices, the *different representations* of the antisymmetric spinorial metric ϵ^{AB}, ϵ_{AB}, $\epsilon^{A'B'}$, and $\epsilon_{A'B'}$ can be used (see Note A.5 and (A.18)). Each of them can be written as the same matrix, given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{A.42}$$

Hence, the spinorial version of the tetrad reads

$$e_\mu^{AA'} = e^a{}_\mu \sigma_a^{AA'} , \qquad e^a{}_\mu = -\sigma^a_{AA'} e_\mu^{AA'} \ . \tag{A.43}$$

Generally, for any *tensor* quantity T defined in a (curved) spacetime, a corresponding spinorial quantity is associated through $\mathsf{T} \to T^{AA'} = e_\mu^{AA'} T^\mu$, with the inverse relation given by $T^\mu = -e^\mu_{AA'} T^{AA'}$.

Equipped with this feature, the foliation of spacetime into spatial hypersurfaces Σ_t from a tetrad viewpoint (*spinorial version*) is straightforward:

- The future-pointing unit normal vector $\mathsf{n} \to n^\mu$ has a spinorial form given by

$$n^{AA'} = e_\mu^{AA'} n^\mu \ . \tag{A.44}$$

- The tetrad (A.43) is thereby decomposed into timelike and spatial components $e_0^{AA'}$ and $e_i^{AA'}$.
- Moreover, the 3-metric is written as

$$h_{ij} = -e_{AA'i} e_j^{AA'} \ . \tag{A.45}$$

  This metric and its inverse are used to lower and raise the spatial indices $i, j, k, \ldots$.
- From the definition of $n^{AA'}$ as a future-pointing unit normal to the spatial hypersurfaces Σ_t, we can further retrieve

$$n_{AA'} e_i^{AA'} = 0 \quad \text{and} \quad n_{AA'} n^{AA'} = 1 \ , \tag{A.46}$$

  which allow us to express $n^{AA'}$ in terms of $e_i^{AA'}$ (see Exercise 4.3).

[15] We henceforth condense and follow the structure in Note A.5 (see Chap. 4).

- Using the lapse function $\mathcal{N}$ and the shift vector $\mathcal{N}^i$, the timelike component of the tetrad can be decomposed according to

$$e_0^{AA'} = \mathcal{N} n^{AA'} + \mathcal{N}^i e_i^{AA'} . \tag{A.47}$$

- Other useful formulas[16] involving the spinorial tetrad $e_i^{AA'}$ with the unit vector in spinorial form $n^{AA'}$ can be found in Sect. A.4.1.

The reader should note that these expressions do indeed allow one to considerably simplify, e.g., the differential equations for the bosonic functionals of the wave function of the universe (see, e.g., Chap. 5).

A.4.1 Metric and Tetrad

Using the above definitions [11], we have the following relations for the timelike normal vector $n_{AA'}$ and the tetrad $e_i^{AA'}$:

$$n_{AA'} n^{AB'} = \frac{1}{2} \varepsilon_{A'}{}^{B'} , \tag{A.48}$$

$$n_{AA'} n^{BA'} = \frac{1}{2} \varepsilon_A{}^B , \tag{A.49}$$

$$e_{AA'i} e_j^{AB'} = -\frac{1}{2} h_{ij} \varepsilon_{A'}{}^{B'} - \mathrm{i}\sqrt{h} \varepsilon_{ijk} n_{AA'} e^{AB'k} , \tag{A.50}$$

$$e_{AA'i} e_j^{BA'} = -\frac{1}{2} h_{ij} \varepsilon_A{}^B - \mathrm{i} \frac{1}{\sqrt{h}} \varepsilon_{ijk} n_{AA'} e^{BA'k} , \tag{A.51}$$

$$e_{AA'i} e^i_{BB'} = n_{AA'} n_{BB'} - \varepsilon_{AB} \varepsilon_{A'B'} . \tag{A.52}$$

From (A.50) and (A.51), by contracting with ε^{ijl}, we obtain

$$n_{AA'} e^{AB'l} = -n^{AB'} e^k_{AA'} = \frac{\mathrm{i}}{2\sqrt{h}} \varepsilon^{ijl} e_{AA'i} e_j^{AB'} , \tag{A.53}$$

$$n_{AA'} e^{BA'l} = -n^{BA'} e^k_{AA'} = -\frac{\mathrm{i}}{2\sqrt{h}} \varepsilon^{ijl} e_{AA'i} e_j^{BA'} . \tag{A.54}$$

In addition, we have

$$g^{\mu\nu} e_\mu^{AA'} e_\nu^{BB'} = -\varepsilon^{AB} \varepsilon^{A'B'} , \tag{A.55}$$

$$g_{\mu\nu} = -e_\mu^{AA'} e_\nu^{BB'} \varepsilon_{AB} \varepsilon_{A'B'} , \tag{A.56}$$

[16] Equations (A.44), (A.45), (A.46), and (A.47) can be contrasted with (A.60), (A.61), (A.62), (A.63), (A.64), and (A.65).

$$2e_{AA'(\mu}e_{\nu)}^{BA'} = g_{\mu\nu}\varepsilon_A{}^B \; , \tag{A.57}$$

$$2e_{AA'(\mu}e_{\nu)}^{AB'} = g_{\mu\nu}\varepsilon_{A'}{}^{B'} \; . \tag{A.58}$$

The normal projection of the expression $\epsilon^{ilm} D_B{}^{B'}{}_{mj} D^C{}_{A'kl}$, used throughout Chap. 4 of Vol. II, is given by [16]

$$E_{Bjk}^{CAB'i} \equiv n^{AA'}\epsilon^{ilm} D_B{}^{B'}{}_{mj} D^C{}_{A'kl} = \frac{-2\mathrm{i}}{\sqrt{h}}\epsilon^{ilm} D_B{}^{B'}{}_{mj} e_l^{CD'} e_{DD'k} n_{A'}^D n^{AA'}$$

$$= \frac{2\mathrm{i}}{\sqrt{h}}\epsilon^{ilm} D_B{}^{B'}{}_{mj} e_l^{CD'} e_{D'k}^A \; . \tag{A.59}$$

Now, regarding the 4-spinor framework in Sects. 4.1.1 and 4.1.2,

$$\mathcal{N}_i = e_{0a}e^a{}_i \; , \tag{A.60}$$

$$\mathcal{N} = \mathcal{N}_j\mathcal{N}^j - e_{0a}e^a{}_0 \; , \tag{A.61}$$

$$e_0{}^a = -\frac{n^a}{\mathcal{N}} \; , \tag{A.62}$$

$$e_{0a} = \mathcal{N} n_a + \mathcal{N}^m e_{ma} \; , \tag{A.63}$$

$$n^\gamma e_{m\gamma} = 0 \; , \tag{A.64}$$

$$\varepsilon^{ijk}\gamma_s\gamma_i = 2\mathrm{i}\gamma^\perp\sigma^{ik} \; , \tag{A.65}$$

and we have the notation

$$-A_\perp = A^\perp = -n^\mu A_\mu = \mathcal{N} A^0 = \frac{1}{\mathcal{N}}\left(A_0 - \mathcal{N}^i A_i\right) \; , \tag{A.66}$$

$$A^{\hat{i}} \equiv A^i - \frac{\mathcal{N}^i}{\mathcal{N}} A^\perp \; . \tag{A.67}$$

A.4.2 Connections and Torsion

In the *second order* formalism applied to $N = 1$ SUGRA [10], the tetrad $e_\mu{}^{CD'}$ and gravitinos $\psi_\mu{}^C$, $\psi_\nu{}^{D'}$ determine the explicit form of the connections.

Four-Dimensional Spacetime

We will use

$$\omega_\mu^{ab} = \omega_\mu^{[ab]} \;\longrightarrow\; \omega_\mu^{AA'BB'} = \omega_\mu^{AB}\varepsilon^{A'B'} + \overline{\omega}_\mu^{A'B'}\varepsilon^{AB} \; , \tag{A.68}$$

where

$$\omega_\mu^{AB} = \omega_\mu^{(AB)} = \mathfrak{w}_\mu^{AB} + \kappa_\mu^{AB} \; , \tag{A.69}$$

while ${}^{\mathrm{s}}\omega_\mu^{AB}$ is the spinorial version of the torsion-free connection form

$$ {}^{\mathrm{s}}\omega_\mu^{ab} = e^{a\nu}\partial_{[\mu}e^b_{\nu]} - e^{b\nu}\partial_{[\mu}e^a_{\nu]} - e^{a\nu}e^{b\rho}e_{c\mu}\partial_{[\nu}e^c_{\rho]} \ , \tag{A.70} $$

and $\kappa_\mu^{AB} = \kappa_\mu^{(AB)}$ is the spinorial version of the contorsion tensor $\kappa_\mu^{ab} = \kappa_\mu^{[ab]}$, with

$$ \kappa_\mu^{AA'BB'} = e_\mu^{AA'} e_\nu^{BB'} \kappa_\mu^{\nu\rho} = \kappa_\mu^{AB}\varepsilon^{A'B'} + \overline{\kappa}_\mu^{A'B'}\varepsilon^{AB} \ . \tag{A.71} $$

Furthermore, explicitly, it will in this case relate to the (4D spacetime) torsion through

$$ \xi_{\mu\nu}^{AA'} = -4\pi \mathrm{i} G \overline{\psi}^{A'}_{[\mu}\psi^A_{\nu]} \ , \tag{A.72} $$

whose tensorial version is

$$ \xi^\rho{}_{\mu\nu} = -e^\rho_{AA'}\xi^{AA'}_{\mu\nu} \ . \tag{A.73} $$

The contorsion tensor κ is defined by

$$ \kappa_{\mu\nu\rho} = \xi_{\nu\mu\rho} + \xi_{\rho\nu\mu} + \xi_{\mu\nu\rho} \ . \tag{A.74} $$

Spatial Representation

Within a 3D (spatial surface) representation, the novel ingredient is the expansion into $n^{AA'}$ and $e_i^{AA'}$. In more detail, the spin connection is written in the form

$$ {}^{(3)}\omega_i^{AA'BB'} = {}^{(3)\mathrm{s}}\omega_i^{AA'BB'} + {}^{(3)}\kappa_i^{AA'BB'} \ , \tag{A.75} $$

and then decomposed into primed and unprimed parts:

$$ {}^{(3)}\omega_i^{AA'BB'} = {}^{(3)}\omega_i^{AB}\overline{\epsilon}^{A'B'} + {}^{(3)}\overline{\omega}_i^{A'B'}\epsilon^{AB} \ . \tag{A.76} $$

Using the antisymmetry ${}^{(3)}\omega_i^{AA'BB'} = -{}^{(3)}\omega_i^{BB'AA'}$, we then obtain the symmetries ${}^{(3)}\omega_i^{AB} = {}^{(3)}\omega_i^{BA}$ and ${}^{(3)}\overline{\omega}_i^{A'B'} = {}^{(3)}\overline{\omega}_i^{B'A'}$ and the explicit representations

$$ {}^{(3)}\omega_i^{AB} = \frac{1}{2}\,{}^{(3)}\omega^A{}_{B'i}{}^{BB'} \ , \qquad {}^{(3)}\overline{\omega}_i^{A'B'} = \frac{1}{2}\,{}^{(3)}\omega_{Bi}{}^{A'BB'} \ . \tag{A.77} $$

Analogous relations hold for ${}^{(3)\mathrm{s}}\omega_i^{AA'BB'}$ and ${}^{(3)}\kappa_i^{AA'BB'}$. Furthermore, we now have

$$ {}^{(3)}\omega_i^{AB} = {}^{(3\mathrm{s})}\omega_i^{AB} + {}^{(3)}\kappa_i^{AB} \ , \tag{A.78} $$

where ${}^{(3\mathrm{s})}\omega_i^{AB}$ is the spinorial version of the spatial torsion-free connection form

$$^{(3s)}\omega_i^{ab} = \left(e^{bj}\partial_{[j}e^a_{i]} - \frac{1}{2}e^{aj}e^{bk}e^c_i\partial_j e_{ck} - \frac{1}{2}e^{aj}n^b n^c\partial_j e_{ci} - \frac{1}{2}n^a\partial_i n^b\right) - (a \leftrightarrow b) \ , \tag{A.79}$$

and κ_i^{AB} is the spinorial version of the spatial contorsion tensor[17] κ_i^{ab}, with

$$^3\kappa_{AA'BB'i} = e^j_{AA'}e^k_{BB'}\,{}^3\kappa_{jki} = {}^3\kappa^{ABi}\varepsilon^{A'B'} + {}^3\overline{\kappa}^{A'B'i}\varepsilon^{AB} \ . \tag{A.81}$$

The three-dimensional torsion-free spin connection $^{(3)s}\omega_i^{AA'BB'}$ can therefore be expressed in terms of $n^{AA'}$ and $e_i^{AA'}$ as

$$\begin{aligned}
^{(3)s}\omega_i^{AA'BB'} &= e^{BB'j}\partial_{[j}e^{AA'}_{i]} \\
&-\frac{1}{2}\left(e^{AA'j}e^{BB'k}e^{CC'}_i\partial_j e_{CC'k} + e^{AA'j}n^{BB'}n^{CC'}\partial_j e_{CC'i} + n^{AA'}\partial_i n^{BB'}\right) \\
&-e^{AA'j}\partial_{[j}e^{BB'}_{i]} \\
&+\frac{1}{2}\left(e^{BB'j}e^{BB'k}e^{CC'}_i\partial_j e_{CC'k} + e^{BB'j}n^{AA'}n^{CC'}\partial_j e_{CC'i} + n^{BB'}\partial_i n^{AA'}\right) \ .
\end{aligned} \tag{A.82}$$

A.4.3 Covariant Derivatives

Still referring to a second order formalism, the tetrad $e_\mu{}^{CD'}$ and the gravitinos $\psi_\mu{}^C$, $\psi_\nu{}^{D'}$ constitute the action variables, with which a *specific* covariant derivative D_μ is associated, acting only on spinor indices (not spacetime, with which Γ would be associated).

Four-Dimensional Spacetime

To be more precise, we use[18]

$$D_\mu e_\nu^{AA'} = \partial_\mu e_\nu^{AA'} + \omega^A_{B\mu}e_\nu^{BA'} + \overline{\omega}^{A'}_{B'\mu}e_\nu^{AB'} \ , \tag{A.83}$$

$$D_\mu \psi_\nu^A = \partial_\mu\psi_\nu^A + \omega^A_{B\mu}\psi_\nu^B \ , \tag{A.84}$$

with $\omega^A_{B\mu}$ the connection form (spinorial representation) as described in Sect. A.4.2 above.

[17] The 3D contorsion is simply obtained by restricting the 4D quantity:

$$^{(3)}\kappa_{ijk} = \kappa_{ijk} \ , \tag{A.80}$$

with the spinorial contorsion $^{(3)}\kappa^{AA'BB'i} = e^{AA'j}e_k^{BB'}\,{}^{(3)}\kappa_{jki} = -\,{}^{(3)}\kappa^{BB'AA'i}$.

[18] Notice that $D_{[\mu}e^{AA'}_{\nu]} = \xi^{AA'}_{\mu\nu}$, where $\xi^{AA'}_{\mu\nu}$ is the spinor version of the torsion.

Spatial Representation

Subsequently, we have the spatial covariant derivative ${}^{(3)}D_j$ acting on the spinor indices, where, for a generic tensor in spinorial form,[19]

$$ {}^{(3)}D_j T^{AA'} = \partial_j T^{AA'} + {}^{(3)}\omega^A_B T^{BA'} + {}^{(3)}\overline{\omega}^{A'}_{B'} T^{AB'} , \tag{A.85} $$

with ${}^{(3)}\omega^A_B$ and ${}^{(3)}\overline{\omega}^{A'}_{B'}$ the two parts of the spin connection (see (A.77)), using the decomposition

$$ {}^{(3)}\omega_i^{AA'BB'} = {}^{(3)\mathrm{s}}\omega_i^{AA'BB'} + {}^{(3)}\kappa_i^{AA'BB'} . \tag{A.86} $$

A.4.4 (Gravitational) Canonical Momenta

Related to the presence of (covariant) derivatives in the action of $N = 1$ SUGRA, we retrieve the momenta conjugate to the tetrad $e_\mu^{CD'}$ and gravitinos ψ_μ^C, $\psi_\nu^{D'}$. The latter require the use of Dirac brackets, which are discussed elsewhere (see Appendix B and Chap. 4), so we present here some elements regarding the former.

For the gravitational canonical momentum, we can write

$$ p^i_{AA'} = \frac{\delta S_{N=1}}{\delta e_i^{AA'}} \longrightarrow p^{ji} = -e^{AA'j} p^i_{AA'} , \qquad p^{\perp i} = n^{AA'} p^i_{AA'} , \tag{A.87} $$

where we can also use (as in the pure gravitational sector, see Sect. 4.2)

$$ p^{(ij)} = -2\pi^{ij} . \tag{A.88} $$

This then leads us to the issue of curvature.

A.4.5 Curvature

In fact, we can express the curvature (e.g., the intrinsic curvature or second fundamental form K_{ij}) in spinorial terms and through the (gravitational) canonical momentum $p^i_{AA'}$, with the assistance of (A.87) and (A.88). Let us recall that[20]

$$ \begin{aligned} \pi^{ij} &\sim -\frac{h^{1/2}}{2\mathsf{k}^2}\left\{K^{(0)(ij)} - \left[\mathrm{tr}K^{(0)}\right]h^{ij}\right\} \\ &\sim -\frac{h^{1/2}}{2\mathsf{k}^2}\left\{\left[K^{(ij)} - \tau^{(ij)}\right] - h^{ij}(K-\tau)\right\} , \end{aligned} \tag{A.89} $$

$$ K_{ij} = -e_i^a \partial_j n_a + n_a \omega_j^{ab} e_{bi} , \tag{A.90} $$

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[19] $T^{AA'\dots ZZ'} = e_\mu^{AA'} \dots e_\nu^{ZZ'} T^{\mu\dots\nu}$.

[20] $p^{[ij]}$ and additional terms in $p^{\perp i}$ will appear in the Lorentz constraint $J_{ab} \leftrightarrow J_{AB}\varepsilon_{A'B'} + J_{A'B'}\varepsilon_{AB}$.

but now emphasizing the influence of the torsion:

$$K_{(ij)} = \frac{1}{2\mathcal{N}}\left(\mathcal{N}_{iIj} + \mathcal{N}_{j|i} - h_{ij,0}\right) - 2\xi_{(ij)\perp} \; , \tag{A.91}$$

$$K_{[ij]} = \xi_{\perp ij} \equiv n^{\mu}\xi_{\mu ij} \; . \tag{A.92}$$

Four-Dimensional Spacetime

The spinor-valued curvature is given by

$$\mathcal{R}^{AB}_{\mu\nu} = \mathcal{R}^{(AB)}_{\mu\nu} = 2\left(\partial_{[\mu}\omega^{AB}_{\nu]} + \omega^{A}_{C[\mu}\omega^{CB}_{\nu]}\right) \; , \tag{A.93}$$

$$\mathcal{R} = e^{\mu}_{AA'}e^{A'\nu}_{B}\mathcal{R}^{AB}_{\mu\nu} + e^{A'\mu}_{A}e^{AB'\nu}\overline{\mathcal{R}}^{A'B'}_{\mu\nu} \; . \tag{A.94}$$

Spatial Representation

The components of the 3D curvature in terms of the spin connection read

$$\begin{aligned} {}^{(3)}R^{AB}_{ij} &= 2\left[\partial_{[i}\,{}^{(3)}\omega^{AB}_{j]} + {}^{(3)}\omega^{A}_{C[i}\,{}^{(3)}\omega^{CB}_{j]}\right] \; , \\ {}^{(3)}\overline{R}^{A'B'}_{ij} &= 2\left[\partial_{[i}\,{}^{(3)}\overline{\omega}^{A'B'}_{j]} + {}^{(3)}\overline{\omega}^{A'}_{C'[i}\,{}^{(3)}\overline{\omega}^{C'B'}_{j]}\right] \; . \end{aligned} \tag{A.95}$$

Because of the symmetry of ${}^{(3)}\omega^{[AB]}_{i} = 0$ and ${}^{(3)}\overline{\omega}^{[A'B']}_{i} = 0$, the chosen notation ${}^{(3)}\omega^{A}_{Bi}$ and ${}^{(3)}\overline{\omega}^{A'}_{B'i}$ is unambiguous. The horizontal position of the indices does not need to be fixed. The scalar curvature is given by

$${}^{(3)}R = e^{i}_{AA'}e^{j}_{BB'}\left[{}^{(3)}R^{AB}_{ij}\overline{\epsilon}^{A'B'} + {}^{(3)}\overline{R}^{A'B'}\epsilon^{AB}\right] \; . \tag{A.96}$$

The same procedure performed on ${}^{(3)\mathrm{s}}\omega^{AA'BB'}_{i}$ leads to the torsion-free scalar curvature:

$$\begin{aligned} {}^{(3\mathrm{s})}R^{AB}_{ij} &= 2\left[\partial_{[i}\,{}^{(3)\mathrm{s}}\omega^{AB}_{j]} + {}^{(3)\mathrm{s}}\omega^{A}_{C[i}\,{}^{(3)\mathrm{s}}\omega^{CB}_{j]}\right] \; , \\ {}^{(3\mathrm{s})}\overline{R}^{A'B'}_{ij} &= 2\left[\partial_{[i}\,{}^{(3\mathrm{s})}\overline{\omega}^{A'B'}_{j]} + {}^{(3\mathrm{s})}\overline{\omega}^{A'}_{C'[i}\,{}^{(3\mathrm{s})}\overline{\omega}^{C'B'}_{j]}\right] \; , \end{aligned} \tag{A.97}$$

and

$${}^{(3\mathrm{s})}R = e^{i}_{AA'}e^{j}_{BB'}\left[{}^{(3\mathrm{s})}R^{AB}_{ij}\epsilon^{A'B'} + {}^{(3\mathrm{s})}\overline{R}^{A'B'}\epsilon^{AB}\right] \; . \tag{A.98}$$

A.4.6 Decomposition with Four-Component Spinors

We now present, in a somewhat summarized manner, a few helpful formulas for the 3 + 1 decomposition of a Cartan–Sciama–Kibble (CSK) theory, and in particular, Einstein gravity with torsion (see Chap. 2), using 4-component spinors [14]:

- The torsion is given by

$$\xi_{\mu\nu}{}^{\lambda} = \frac{1}{2}\left(\Gamma_{\mu\nu}{}^{\lambda} - \Gamma_{\nu\mu}{}^{\lambda}\right) , \tag{A.99}$$

where $\mathbf{e}_\lambda$ are basis vectors, which we will take as a coordinate basis with e_i tangent to a spacelike hypersurface and the normal n with components $\mathsf{n} \to n_\mu = (-\mathcal{N}, 0, 0, 0)$, and where the components of the metric are as in (2.6), (2.7), and (2.8) of Chap. 2. Then we can write

$$\mathbf{e}_{\mu;\nu} = \mathbf{e}_\lambda \Gamma_{\mu\nu}{}^{\lambda} , \tag{A.100}$$

$$\Gamma_{\mu\nu}{}^{\lambda} = \Gamma^{(0)\lambda}_{\mu\nu} - \kappa_{\mu\nu}{}^{\lambda} , \tag{A.101}$$

$$\kappa_{\mu\nu}{}^{\lambda} = \xi_{\mu\nu}{}^{\lambda} - \xi_{\mu}{}^{\lambda}{}_{\nu} + \xi^{\lambda}{}_{\mu\nu} , \tag{A.102}$$

with $\Gamma^{(0)\lambda}_{\mu\nu}$ the Christoffel symbols and $\kappa_{\mu\nu}{}^{\lambda}$ the contorsion tensor.
- The extrinsic curvature K_{ij} can then be computed and retrieved from $\mathsf{n}_{;j}$ by (see (2.9))

$$K_{ji} = -\mathcal{N}\,{}^{(4)}\Gamma^0_{ji} = \frac{1}{2\mathcal{N}}\left(-h_{ji,0} + \mathcal{N}_{j|i} + \mathcal{N}_{i|j}\right) - \kappa_{ji\perp} , \tag{A.103}$$

$$K_{(ji)} = \frac{1}{2\mathcal{N}}\left(-h_{ji,0} + \mathcal{N}_{j|i} + \mathcal{N}_{j|i}\right) + \tau_{(ji)} , \tag{A.104}$$

$$K_{[ji]} = \xi_{ij\perp} , \qquad \tau_{ji} \equiv 2\xi_{j\perp i} , \tag{A.105}$$

where | denotes the derivative involving the Christoffel symbols for h_{ij}.
- The curvature is then obtained from

$${}^{(4)}R = {}^{(3)}R - K_{ij}K^{ij} + K^2 - 2\,{}^{(4)}R_{\perp}{}^{\alpha}{}_{\perp\alpha} , \tag{A.106}$$

$${}^{(4)}R_{\perp}{}^{\alpha}{}_{\perp\alpha} = \left(n^\gamma n^\beta_{\wr\alpha} - n^\beta n^\gamma_{\wr\gamma}\right)_{\wr\beta} - K^{ij}K_{ij} + K^2 1 - 2\xi^{\alpha\beta}{}_{\perp} n_{\alpha\wr\beta} . \tag{A.107}$$

- Finally, or almost, the Lagrangian density is

$$\sqrt{h}L = \mathcal{N}h^{1/2}\left[{}^{(3)}R - K_{ij}K^{ij} + K^2 - 2\left(n^\gamma n^\beta_{\wr\alpha} - n^\beta n^\gamma_{\wr\gamma}\right)_{\wr\beta} + 4\xi^{\alpha\beta}{}_{\perp} n_{\alpha\wr\beta}\right] , \tag{A.108}$$

from which the first two lines of (4.17) are obtained.

From the above, the Hamiltonian and constraints are[21]

$$\mathcal{H}_m = -2h_{mi}\pi^{ik}_{\ |k} \ , \tag{A.109}$$

$$\mathcal{J}^{ab} \equiv p^{ka}e^b_k - p^{kb}e^a_k \ , \tag{A.110}$$

$$\mathcal{H}_\perp \sim h^{-1/2}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - h^{1/2\ (3)}R + h^{1/2}\left[\tau^{(ij)}\tau_{(ij)} - \tau^2\right] \tag{A.111}$$
$$+\sqrt{h}\xi_{ij\perp}\xi^{ij}{}_\perp - 2h^{1/2}\tau_{[ij]}\xi^{ji}{}_\perp + \sqrt{h}q^k\rho_k - 2h^{1/2}\rho^i_{||\,i} + 2h^{1/2}\rho^i\rho_i \ ,$$

$$H = \mathcal{N}\mathcal{H}_\perp + \mathcal{N}^i\mathcal{H}_i + \mathcal{M}_{ab}\mathcal{J}^{ab} \ . \tag{A.112}$$

A.4.7 Equations Used in Chap. 4 of Vol. II

We employ several derivatives of functionals with respect to the tetrad $e^{AB'}_j$ [16]. Two of them are:

$$\epsilon^{ilm}n^{AA'}\frac{\delta}{\delta e^{AB'}_j}(D_B{}^{B'}{}_{mj}D^C{}_{A'kl}) \tag{A.113}$$

and

$$n^{AA'}\frac{\delta}{\delta e^{AB'}_j}D^{BB'}{}_{ij} \ . \tag{A.114}$$

First we need an explicit form for $\delta n^{AA'}/\delta e^{BB'}_j$. With (4.109), which expresses $n^{AA'}$ in terms of the tetrad, the relation $n^{AA'}e_{AA'i} = 0$ implies

$$0 = e^{CC'i}\frac{\delta n^{AA'}e_{AA'i}}{\delta e^{BB'}_j} = n^{CC'}n_{AA'}\frac{\delta n^{AA'}}{\delta e^{BB'}_j} - \epsilon_A{}^C\epsilon_{A'}{}^{C'}\frac{\delta n^{AA'}}{\delta e^{BB'}_j} + e^{CC'j}n_{BB'} \ . \tag{A.115}$$

In addition, we have

$$\frac{\delta n^{AA'}}{\delta e^{BB'}_j} = \frac{\delta n^{CC'}n_{CC'}n^{AA'}}{\delta e^{BB'}_j} = 2n_{CC'}n^{AA'}\frac{\delta n^{AA'}}{\delta e^{BB'}_j} + \frac{\delta n^{AA'}}{\delta e^{BB'}_j} \ , \tag{A.116}$$

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[21] The reader should notice that, with the above alone, i.e., no gravitino (matter) action, we have $P^{\mu\nu\lambda} = 0$ for the conjugate to torsion $\xi_{\mu\nu\lambda}$, whose conservation leads to $\xi_{\mu\nu\lambda} = 0$. *But* if the Rarita–Schwinger field is present in an extended action, with Lagrangian terms, e.g., $\varepsilon^{\lambda\mu\nu\rho}\overline{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho$, then

$$\xi_{\mu\nu\lambda} = -\frac{1}{4}\overline{\psi}_\mu\gamma_\lambda\psi_\nu \ ,$$

and torsion *cannot* then be ignored.

whence

$$\frac{\delta n^{AA'}}{\delta e_j^{BB'}} = e^{AA'j} n_{BB'} \ . \tag{A.117}$$

We then use the derivative of the determinant h of the three-metric:

$$\frac{\partial h}{\partial h_{ij}} = h^{ij} h \ . \tag{A.118}$$

Hence,

$$\frac{\delta h}{\delta e_i^{AA'}} = -2h e^i_{AA'} \ . \tag{A.119}$$

Using this and (A.48), (A.49), (A.50), (A.51), and (A.52), we can calculate the expressions

$$\begin{aligned} & n^{AA'} \epsilon^{ilm} \frac{\delta}{\delta e_j^{AB'}} (D_B{}^{B'}{}_{mj} D^C{}_{A'kl}) \\ & = -4 n^{AA'} \epsilon^{ilm} \frac{\delta}{\delta e_j^{AB'}} \left(\frac{1}{h} e_{Bj}^{D'} e_{DD'm} n^{DB'} e_l^{CE'} e_{EE'k} n^E{}_{A'} \right) \\ & = \epsilon_B{}^C \delta^i_k \frac{\mathrm{i}}{\sqrt{h}} \left(1 - 1 + \frac{1}{2} - \frac{1}{2} \right) + \frac{2\mathrm{i}}{\sqrt{h}} \left(2 e^{CB'i} e_{BB'k} + e^i_{BB'} e_k^{CB'} \right) \\ & = \frac{-3\mathrm{i}}{\sqrt{h}} \delta^i_k \epsilon_B{}^C - 2 h^{ij} \epsilon_{jkl} n^{CB'} e^l_{BB'} \ , \end{aligned} \tag{A.120}$$

and

$$\begin{aligned} n^{AA'} \frac{\delta}{\delta e_j^{AB'}} D^{BB'}{}_{ij} &= -2i n^{AA'} \frac{\delta}{\delta e_j^{AB'}} \left(\frac{1}{\sqrt{h}} e_j^{BC'} e_{CC'i} n^{CB'} \right) \\ &= -\frac{2i}{\sqrt{h}} n^{AA'} n^{BC'} e_{AC'i} \ . \end{aligned} \tag{A.121}$$

Finally, we can write a transformation rule to express derivatives in terms of the tetrad $e_i^{AA'}$ as derivatives in terms of the three-metric h_{ij}. Let $\mathcal{F}[e]$ be a functional depending on the tetrad, and note that h_{ij} can be expressed in terms of the tetrad, since we have the relation $h_{ij} = -e_i^{AA'} e_{AA'j}$. Moreover, there is of course no inverse relation. We thus restrict the functional $\mathcal{F}$ by demanding that it can be written in the form $\mathcal{F}[h_{ij}]$. Consequently, using the chain rule, we find for the transformation of the functional derivatives:

$$\begin{aligned}\frac{\delta \mathcal{F}}{\delta e_i^{AA'}} &= \frac{\delta \mathcal{F}}{\delta h_{jk}} \frac{\delta h_{jk}}{\delta e_i^{AA'}} = -\frac{\delta \mathcal{F}}{\delta h_{jk}} \epsilon_{BC} \epsilon_{B'C'} \frac{\delta e_j^{BB'} e_k^{CC'}}{\delta e_i^{AA'}} \\ &= -\frac{\delta \mathcal{F}}{\delta h_{ik}} \epsilon_{AC} \epsilon_{A'C'} e_k^{CC'} - \frac{\delta \mathcal{F}}{\delta h_{ji}} \epsilon_{BA} \epsilon_{B'A'} e_j^{BB'} \\ &= -2 \frac{\delta \mathcal{F}}{\delta h_{ij}} e_{AA'j} \ . \end{aligned} \tag{A.122}$$

Using $e^{AA'i} e_{AA'j} = -\delta^i_j$, the inverse relation for an arbitrary functional $\mathcal{G}[h_{ij}]$ is simply

$$\frac{\delta \mathcal{G}}{\delta h_{ij}} = \frac{1}{2} e^{AA'j} \frac{\delta \mathcal{G}}{\delta e_i^{AA'}} \ . \tag{A.123}$$

Note that it is always possible to rewrite $\mathcal{G}[h_{ij}]$ in the form $\mathcal{G}[e]$.

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Appendix B
Quantization of Hamiltonian Systems with Constraints

The main content of this appendix summarizes the essential elements presented at the website [1]. In order to be as clear as possible, we should define what we mean by constraints. In simple terms, a constraint is a relationship between the degrees of freedom of a system that holds for all times, i.e., restrictions on the dynamics that step in even before the equations of motion are solved. Hence, constraints are not to be mistaken for constants of motion.

Let us consider an example of interest in the context of general relativity and its consistent extension to SUGRA. The Hamiltonian formulation of the equations of motion is intended to indicate how the spatial geometry of the universe changes with time. But the spatial geometry is the geometry of a 3D hypersurface in a 4D manifold, satisfying the Gauss–Codazzi equations *for all times*, i.e., a set of constraints for any time evolution. As pointed out in Chap. 2, this is related to invariance properties under coordinate transformations. This is more than just a coincidence, as we will emphasize below.[1]

B.1 Primary and Secondary Constraints

As a first stage in our exploration, let us take constraints that are *imposed*. In more specific terms, consider an action

$$S\left[q^{\alpha},\dot{q}^{\beta}\right]=\int \mathrm{d}t\ L(q,\dot{q})\ , \tag{B.1}$$

[1] The reader should compare the Hamiltonian analysis of Chap. 2 with that of Maxwell electromagnetism. Like the lapse and shift variables, the time derivative of A_0 does not appear in the action. The conjugate momentum is *always* zero, i.e., it is a *constraint*. In particular, it is not imposed *a priori* and results from the definition of conjugate momenta.

where t is just a parameter, the dot over a symbol represents $\mathrm{d}/\mathrm{d}t$, and the general coordinates ranging from $\alpha = 1$ to n are represented by $q^\alpha(t)$. The recipe for including constraints consistently is as follows:[2]

- The evolution of q and $\dot{q}$ must satisfy m relations of the form $0 = \varphi_a(q, \dot{q})$, where $a = 1, \ldots, m$.
- Change the Lagrangian L by $L(q, \dot{q}) \longrightarrow L^{(1)}(q, \dot{q}; \ell) \equiv L(q, \dot{q}) - \ell^a \varphi_a(q, \dot{q})$, where ℓ^a are the Lagrange multipliers.
- Take generalized coordinates $Q = q \cup \ell \equiv Q^A$, $A = 1, \ldots, n+m$.
- Then we also have $\partial L^{(1)}/\partial \dot{\ell}^a = 0 \implies \varphi_a = 0$.
- Moreover, we also have $p_\alpha = \partial L^{(1)}/\partial \dot{q}^\alpha$ and $\pi_a = \partial L^{(1)}/\partial \dot{\ell}^a = 0$.
- To work with coordinates and momenta (Hamiltonian approach) instead of coordinates and velocities (Lagrangian approach), we need to invert[3] the above equations for p_α and π_a and express $\dot{q}$ in terms of Q and p.
- If we can find all $\dot{q} = \dot{q}(Q, p)$, these are said to be *primarily expressible* velocities.
- However, this obviously does *not* apply to the $\dot{\ell}$, and they are *primarily inexpressible*.
- The Hamiltonian is then $H(q, p) = p_\alpha q^\alpha - L$, which is a function of q, p and not $\dot{q}, \ell$, from which we recover the equations

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha} + \ell^a \frac{\partial \varphi_a}{\partial p_\alpha}\,, \tag{B.2}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} - \ell^a \frac{\partial \varphi_a}{\partial q^\alpha}\,. \tag{B.3}$$

- Motivated by the form of (B.2) and (B.3), the Poisson bracket is defined by

$$[F, G]_{\mathrm{P}} \equiv \frac{\partial F}{\partial q^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial G}{\partial q^\alpha}\frac{\partial F}{\partial p_\alpha}\,.$$

  For any function f, we also write

$$\begin{aligned}\dot{f} &= \frac{\partial f}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha = \left[f, H\right]_{\mathrm{P}} + \ell^a \left[f, \varphi_a\right]_{\mathrm{P}} \\ &= \left[f, H + \ell^a \varphi_a\right]_{\mathrm{P}} - \varphi_a \left[f, \ell^a\right]_{\mathrm{P}} \cong \left[f, H + \ell^a \varphi_a\right]_{\mathrm{P}}\,,\end{aligned} \tag{B.4}$$

  where the last term arises by *imposing the constraint after* calculating the expression. Hence the symbol $\cong$, constituting the *weak equality sign*, as introduced by Dirac. The Poisson brackets must be worked out *before* any constraints are imposed.

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[2] Here $\alpha, \ldots,$ are just indices identifying different coordinates in configuration space, a identifies the constraint (number), and A is then associated with a larger configuration space. In fact, $\{A\} \equiv \{a\} \oplus \{\alpha\}$.

[3] Assuming that the Jacobian of the transformation from $\dot{q}$ to p is non-zero.

B.1.1 Primary Constraints

Two questions spring to mind at this point. First, the Lagrange multipliers are still undetermined. Is this all that can be said about them? And, secondly, are the constraints themselves conserved in time? Without this, it would be difficult to proceed consistently with the framework in this section.

By way of a small (apparent) detour, let us define the 'mass' matrix by

$$M_{AB} \equiv \frac{\partial^2 L^{(1)}}{\partial \dot{Q}^A \partial \dot{Q}^B} ,$$

and consider the above description (B.1), (B.2), (B.3), and (B.4) as if we had ab initio coordinate variables Q^A and momenta P_A. Retrieving the Hamiltonian from

$$H(Q, P) = P_A \dot{Q}^A - L(Q, \dot{Q}) ,$$

we still need to find $\dot{Q} = \dot{Q}(Q, P)$, transforming variables from P to $\dot{Q}$. This requires

$$0 \neq \det \frac{\partial P^A}{\partial \dot{Q}^B} = \det \frac{\partial^2 L^{(1)}}{\partial \dot{Q}^A \partial \dot{Q}^B} = \det M_{AB} .$$

These Lagrangians are non-singular, i.e., we have the same situation as in the case of (B.2) and (B.3), but for

$$L^{(1)}(q, \dot{q}; \ell) = L(q, \dot{q}) - \ell^a \varphi_a(q, \dot{q})$$

we have $\det M_{AB} = 0$ and the Lagrangian is singular. This is caused by the presence of constraints in the theory. In other words, singular Lagrangians give rise to constraints that emerge fully in the Hamiltonian formulation. It is *impossible* to express *all* the velocities as functions of the coordinates and momenta, although it may be possible to express *some* of them, i.e., those that are primarily expressible.

Subsequently, for a *generic* case, let us divide them up according to

$$Q = q \cup \ell , \tag{B.5}$$
$$\dot{Q} = \dot{q} \cup \dot{\ell} , \tag{B.6}$$
$$P = p \cup \pi , \tag{B.7}$$

where $\dot{q}$ are the primarily expressible velocities and $\dot{\ell}$ the primarily *in*expressible velocities. The Hamiltonian can then be written

$$H(Q, P) = p_\alpha \dot{q}^\alpha (Q, P) + \pi_a \dot{\ell}^a - L\left[Q, \dot{q}^\alpha (Q, P), \dot{\ell}\right] .$$

Hence,

$$\left.\frac{\partial H(Q,P)}{\partial \dot{\ell}^b}\right|_{\pi_a} = 0 = \pi_b - \frac{\partial L}{\partial \dot{\ell}^b} \implies \left.\frac{\partial^2 H(Q,P)}{\partial \dot{\ell}^b \partial \dot{\ell}^c}\right|_{\pi_a} = \frac{\partial^2 L}{\partial \dot{\ell}^b \partial \dot{\ell}^c} = 0 \;, \quad \text{(B.8)}$$

and $\partial L/\partial \dot{\ell}^b$ is independent of $\dot{\ell}$. Defining

$$f_a(Q,P) \equiv \frac{\partial L}{\partial \dot{\ell}^b} \;, \quad \text{(B.9)}$$

we write

$$0 = \varphi^{(1)}(Q,P) = \pi_a - f_a(Q,P) \;. \quad \text{(B.10)}$$

The relation (B.10) between the coordinates and momenta holds for all times, thereby constituting constraints, whose number is equal to the number of primarily inexpressible velocities. These are thus designated as *primary* constraints.

Furthermore, singular Lagrangian theories possess primary constraints relating the coordinates and momenta for all times. Let us be more precise. The equations $\pi_a = 0$ constitute primary constraints (see Sect. 2.2.2). Finally, with the label (1) assigned to primary constraints and using υ instead of ℓ, we can summarize by

$$\dot{f} \cong \left[f, H + \upsilon^a \varphi_a^{(1)}\right]_{\mathrm{P}} \;. \quad \text{(B.11)}$$

But we still need to discuss (a) whether the Lagrange multipliers are to be undetermined and (b) how to ensure that the (primary) constraints like $\varphi_a^{(1)}$ are conserved.

B.1.2 Secondary Constraints

Let us then apply (B.11) to $\varphi_a^{(1)}(Q,P)$. It follows that

$$\dot{\varphi}_b^{(1)} \cong \left[\varphi_b^{(1)}, H\right]_{\mathrm{P}} + \upsilon^a \left[\varphi_b^{(1)}, \varphi_a^{(1)}\right]_{\mathrm{P}} \;, \quad \text{(B.12)}$$

and $\dot{\varphi}_b^{(1)} \cong 0$ implies an equation the primary constraints must satisfy. If it happens that $\left[\varphi_b^{(1)}, \varphi_a^{(1)}\right]_{\mathrm{P}} \cong 0$, then $\left[\varphi_b^{(1)}, H\right]_{\mathrm{P}} \cong 0$, and if $\left[\varphi_b^{(1)}, H\right]_{\mathrm{P}}$ does *not* vanish when the constraints are imposed, then $\left[\varphi_b^{(1)}, H\right]_{\mathrm{P}}$ constitutes *another* constraint, referred to as a *secondary* constraint. These are denoted by $\varphi_b^{(2)}$. Furthermore,

$$\dot{f} \cong \left[f, H + \upsilon^I \varphi_I\right]_{\mathrm{P}} \;, \quad \text{(B.13)}$$

$$\varphi_I \equiv \varphi_a^{(1)} \cup \varphi_b^{(2)} \;. \quad \text{(B.14)}$$

Repeating this process we may or may not find more constraints at different stages, and proceeding in this way, we will eventually find

$$\dot{f} \cong \left[f, H_{\mathrm{T}}\right]_{\mathrm{P}} \ , \tag{B.15}$$

where H_{T} is the *total* Hamiltonian (see Sect. 2.2.2) and the index I runs over all the constraints:

$$H_{\mathrm{T}} = H + \upsilon^I \varphi_I \ . \tag{B.16}$$

B.2 First and Second Class Constraints

Once we have obtained the complete set of constraints (both primary and secondary) and the total Hamiltonian, the demand that the constraints have zero time derivative will allow us to address the issue of the υ^I quantities, i.e., we will be able to determine them. To be more concrete, this means

$$0 \cong [\varphi_I, H]_{\mathrm{P}} + \upsilon^J \Delta_{IJ} \ , \tag{B.17}$$

$$\Delta_{IJ}(Q, P) \cong [\varphi_I, \varphi_J]_{\mathrm{P}} \ , \tag{B.18}$$

a linear equation from which the υ^I can be determined. There are then two possibilities.

The Case $\det \Delta \neq 0$

- In this case, after finding $\Delta^{-1}_{IJ} = \Delta^{IJ}$, we obtain $\upsilon^J \cong -\Delta^{IJ}\,[\varphi_J, H]_{\mathrm{P}}$.
- We introduce the *Dirac bracket*

$$[F, G]_{\mathrm{D}} \equiv [F, G]_{\mathrm{P}} - [F, \varphi_I]_{\mathrm{P}}\, \Delta^{IJ}\, [\varphi_J, G]_{\mathrm{P}} \ . \tag{B.19}$$

- We then have

$$\dot{f} \cong \left[f, H\right]_{\mathrm{P}} - \left[f, \varphi_I\right]_{\mathrm{P}} \Delta^{IJ}\, [\varphi_J, H]_{\mathrm{P}} \equiv \left[f, H\right]_{\mathrm{D}} \ . \tag{B.20}$$

The Case $\det \Delta \cong 0$

- This is where interesting additional features appear. In particular, the concept of *first* and *second class* contraints emerges.
- We can transform the original set of constraints using some matrix $\boldsymbol{\Gamma}$ with $\breve{\varphi}_I = \boldsymbol{\Gamma}^I_{\ J} \varphi_I$, getting the matrix in a diagonal form

$$\breve{\Delta} \equiv \left[\breve{\Delta}_{IJ}\right] = \begin{bmatrix} \boldsymbol{\Upsilon} & \\ & \boldsymbol{O} \end{bmatrix} , \tag{B.21}$$

where $\boldsymbol{\Upsilon}$ is an antisymmetric $R \times R$ matrix with $\det \boldsymbol{\Upsilon} \neq 0$, whereas $\breve{\Delta}$ is a $D \times D$ matrix and O is $N \times N$, with $D = R + N$.

- For O, we have $\xi_r^I(Q, P)$ satisfying

$$0 \cong \xi_r^I(Q, P)\Delta_{IJ} \ , \quad r = 1, \ldots, N \ ,$$

where $\xi_r^I(Q, P) = \delta_{r+R}^I$ constitute the linearly independent set of (null) eigenvectors of $\breve{\Delta}$.

- A solution of $0 \cong \breve{\Delta}\upsilon + b$ (see (B.17) and (B.18)) is

$$\upsilon^I \cong \vartheta^I + \varpi^r \xi_r^I \ , \tag{B.22}$$

where ϑ^I satisfies $\vartheta^J \breve{\Delta}_{IJ} \cong -[\varphi_I, H]_{\mathrm{P}}$ and ϖ^r are *totally arbitrary*.[4]

- Hence, from (B.14), (B.15), and (B.16), we write

$$\dot{f} \cong \left[f, \breve{H} + \varpi^r \zeta_r\right]_{\mathrm{P}} \ , \tag{B.23}$$

$$\breve{H} \equiv H + \vartheta^I \varphi_I \ , \tag{B.24}$$

$$\zeta_r \equiv \xi_r^I \varphi_I = \xi_r^I \breve{\varphi}_I = \breve{\varphi}_{r+R} \ . \tag{B.25}$$

- Note that $\left[\varphi_I, \breve{H}\right]_{\mathrm{P}} = 0 = [\varphi_I, \zeta_r]_{\mathrm{P}}$, i.e., $\breve{H}$ and ζ_r commute with all constraints (the Poisson brackets are zero). These are called the *first-class* Hamiltonian and constraints,[5] respectively.
- And there is more. For the first R members of the $\breve{\varphi}$ set, labeled by $r' = 1, \ldots, R$,

$$\chi_{r'} = \delta_{r'}^I \breve{\varphi}_I \ , \tag{B.26}$$

$$\Upsilon_{r's'} = [\chi_{r'}, \chi_{s'}]_{\mathrm{P}} \ , \tag{B.27}$$

and from $\det \Upsilon \cong 0$, we cannot have all the entries in any row or column of Υ vanishing (Dirac like). Hence each element $\chi_{r'}$ will not commute with at least one other. These constitute the so-called *second-class* constraints.

- φ can thus be split into a set of N first-class constraints and R second-class constraints.
- Finally, the first-class Hamiltonian becomes

$$\breve{H} = H - \chi_{r'} \Upsilon^{r's'} [\chi_{s'}, H]_{\mathrm{P}} \ . \tag{B.28}$$

With $[\chi_{s'}, \zeta_r]_{\mathrm{P}} \cong 0$ (see (B.19) and (B.20)), it then follows that

$$\begin{aligned} \dot{f} &= \left[f, H\right]_{\mathrm{P}} - \left[f, \chi_{r'}\right]_{\mathrm{P}} \Upsilon^{r's'} [\chi_{s'}, H]_{\mathrm{P}} + \varpi^r \left[f, \zeta_r\right]_{\mathrm{P}} \\ &= \left[f, H + \varpi^r \zeta_r\right]_{\mathrm{D}} . \end{aligned}$$

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[4] But with a pertinent physical meaning, as we will soon show.

[5] Since $\zeta_r \cong 0$.

Now the time derivative function $\dot{f}$ of Q and P contains N *arbitrary* quantities ϖ^r. The path in configuration (or phase) space is *not* uniquely determined. To extract a rather relevant piece of information, it is important to bear in mind that N is the number of first-class constraints. It so happens that we can take f evolving and depending on our choice of ϖ^r, as long as the real physical situation is the same, i.e., all are physically equivalent. The different possibilities from ϖ^r are related to one another by a *gauge* transformation. So in fact, theories with first-class constraints constitute *gauge* theories, and the generators of gauge transformations are the first-class constraints. To conclude this issue, note that the Poisson bracket of any first-class quantity with any of the constraints is (Dirac like) equal to zero, and then the first-class constraints form a *closed algebra*, as they should for the generators of a group of gauge transformations.

The reader may now be asking why we had to go through several chapters before emphasising the importance of eliminating second-class constraints through the use of the Dirac bracket?

B.3 The Dirac Bracket and Quantization

Quantum mechanically, the commutator between variables $\hat{X}$ (the hat on the variable identifies that it is an operator, i.e., an observable) is taken to be their Poisson bracket evaluated at $\hat{X} = X_{\text{classical}}$:

$$\left[\hat{X}^a, \hat{X}^b\right] = \mathrm{i}\hbar \left[X^a, X^b\right]_{\mathrm{P}; \hat{X}=X} , \tag{B.29}$$

with the rather peculiar feature that, for functions of those observables,[6]

$$\left[F\big(\hat{X}^a\big), G\big(\hat{X}^b\big)\right] = \mathrm{i}\hbar \left[F(X^a), G(X^b)\right]_{\mathrm{P}; \hat{X}=X} + \mathcal{O}(\hbar^2) , \tag{B.30}$$

i.e., in the (semi-)classical limit where $\hbar \to 0$, the $\mathcal{O}(\hbar^2)$ terms can be reasonably ignored.

Let us then consider a system where there are *only* second-class constraints φ_I. For a state Ψ, with each φ_I now understood as an operator and dropping the hat for simplicity, we require

$$\varphi_I \left|\Psi\right\rangle = 0 , \tag{B.31}$$

[6] Only if the variables are ordered in a specific way do we get

$$\left[F\big(\hat{X}^a\big), G\big(\hat{X}^b\big)\right] = \mathrm{i}\hbar \left[F(X^a), G(X^b)\right]_{\mathrm{P}; \hat{X}=X} .$$

Compare with the case $F = x^2$, $G = p^2$, where $\left[x, p\right]_{\mathrm{P}} = 1$. Hence, $\left[x^2, p^2\right]_{\mathrm{P}} = 4xp$ and $\left[\hat{x}^2, \hat{p}^2\right] = \mathrm{i}\hbar 2\,(xp + px)$. This is an example of the *ordering ambiguity* that exists when proceeding from classical to quantum mechanical equations.

which will lead to $\langle F(\varphi)\rangle = 0$, *for any function* F of φ_I, and at *all times*. Quantum mechanically, this implies, with the usual commutator,

$$[\varphi_I, \varphi_J]\,|\Psi\rangle = 0\ , \tag{B.32}$$

which is *classically*

$$[\varphi_I, \varphi_J]_{\mathrm{P}} = 0\ . \tag{B.33}$$

But this is a problem, as φ_I are second class. It is *impossible* to satisfy (B.33) for *all* I and J. The solution is to use Dirac brackets. We thus use instead $[\varphi_I, \varphi_J]_{\mathrm{D}} = 0$, i.e., Dirac brackets rather than Poisson brackets, when quantizing. Hence, we use

$$\left[\hat{X}^a, \hat{X}^b\right] = \mathrm{i}\hbar \left[X^a, X^b\right]_{\mathrm{D};\,\hat{X}=X} \tag{B.34}$$

and

$$\left[F\big(\hat{X}^a\big), G\big(\hat{X}^b\big)\right] = \mathrm{i}\hbar \left[F(X^a), G(X^b)\right]_{\mathrm{D};\,\hat{X}=X} + \mathcal{O}(\hbar^2)\ , \tag{B.35}$$

from which

$$\left[\varphi_I, G\big(\hat{X}^b\big)\right] = \mathcal{O}(\hbar^2)\ , \tag{B.36}$$

i.e., neglecting $\mathcal{O}(\hbar^2)$, the second-class constraints commute with every function of Q and P. This determines that they are c-numbers, and their action on states is by scalar multiplication $\varphi_I\,|\Psi\rangle = c\,|\Psi\rangle$. Quantum mechanically, this means that we can satisfy the constraints by imposing $c = 0$, i.e., we can set $\varphi_I = 0$. When we quantize with Dirac brackets, the constraints *in this particular case* can be freely set to zero for any operator.

But what happens if we have a different situation, namely a system where *there are also* first-class constraints? It is reasonable to quantize this system, *not* by converting Poisson brackets into commutators, but by working with Dirac brackets from the start. Let us assume that we have our constraints φ as $\varphi = \chi \cup \gamma$, where χ denote second-class constraints and γ the first-class constraints, defining

$$[\chi, G]_{\mathrm{D}} = 0\ , \tag{B.37}$$

$$\left[\gamma_r, \gamma_s\right]_{\mathrm{D}} = \sigma_{rs}^{p}\gamma_p\ . \tag{B.38}$$

Quantization can proceed by adopting (B.34), (B.35), and (B.36), with second-class constraints commuting with everything, and setting their operators equal to zero. Concerning the first-class constraints we have to *impose*

$$\gamma_r\,|\Psi\rangle = 0\ , \tag{B.39}$$

as a restriction on the Hilbert space. State vectors $|\Psi\rangle$ satisfying (B.39) are said to be physical, and the space they span is the physical state space. This is the Dirac quantization procedure for systems with first-class constraints (see Sect. 2.5). But, of course, the operator ordering issues mentioned above remain. Nevertheless, at the classical level, the Dirac quantization procedure[7] implies that the classical quantities equivalent to quantum observables are first-class constraints, which are gauge invariant in the Hamiltonian formalism. Hence, physical states are to be interpreted as gauge invariant quantities.

Reference

1. userweb.port.ac.uk/∼ seahras/documents/reviews/quantization.pdf 289

[7] Another route here is to fix the gauge classically, and then quantize the system. We thus eliminate a gauge freedom by adding more constraints to our system in an ad hoc manner. The number of supplementary constraints is equal to the number of first-class constraints. In more precise terms, we transform the system from a first-class to an *entirely* second-class scenario, obtaining a *new* matrix det $\Delta \neq 0$ and then solving to determine the specific form of the extra constraints. Assuming this has been achieved, i.e., transforming our classical gauge system into a system with only second-class constraints, we go on to find the Dirac brackets and quantize as indicated above, with the commutators given by Dirac brackets and the complete set of constraints as operator identities.

But is this quantization procedure equivalent to the Dirac method? In fact, there is *no* general proof that Dirac quantization is equivalent to canonical quantization (gauge fixing) for first-class systems. So here is another ambiguity in the quantization procedure. This is the case for all theories that are invariant under transformations of the time parameter t, viz., general relativistic descriptions of physics with this type of symmetry, and others, like SUSY, leading to SUGRA theories and thus SQC.

Appendix C
Solutions to Exercises

Problems of Chap. 2

2.1 Canonical Form of General Relativity, *Vierbein* and the Lorentz Constraint

General relativity is usually presented using the metric tensor to represent the basic field variables. This is quite appropriate for all bosonic theories of gravitation. However, when dealing with torsion and more precisely with spinors, a larger set of variables for the geometry is mandatory.[1] To be more precise, the tetrad representation allows one to describe local orthonormal Lorentz frames at each point in space time, where spinors can be adequately formulated [1].

We may consider a tetrad field as a set of four orthogonal vector fields, e.g., $e^a(x^\nu)$, $a = (0, 1, 2, 3)$ labelling the independent vectors, to which a coordinate index μ is assigned as $e^a{}_\mu$. Therefore,

$$e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu} , \qquad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab} .$$

Let us then consider the action in canonical form:

$$S \sim \int \mathrm{d}^4x \left[\pi^{ij}\dot{h}_{ij} - \mathcal{N}_\mu \mathcal{H}^\mu(\pi^{lm}, h_{rs})\right] , \tag{C.1}$$

associated with *six* pairs of conjugate spatial variables (π^{lm}, h_{rs}), $l, m = 1, 2, 3$, the Lagrange multipliers $\mathcal{N}_\mu$, and the constraints $\mathcal{H}^\mu$. Then using

$$h_{ij} = e_{ia}e^a_j \implies \dot{h}_{ij} = e_{ia}\dot{e}^a_j + \dot{e}_{ia}e^a_j , \tag{C.2}$$

[1] See Sects. 3.4 and A.3. Moreover, spinors (fermions) can only be properly defined in the tangent space of the spacetime (manifold), associated with the transformation properties of the Lorentz group, and *not* with those of the general coordinate transformation group. Hence the need for Lorentz indices or a tetrad structure to project down to the tangent space.

the canonical action using the vierbein (tetrad) can be taken as

$$S \sim \int \mathrm{d}^4x \left\{ \pi^{ia}\dot{e}_{ia} - \mathcal{N}_\mu(e_{\mu a})\mathcal{H}^\mu[\pi^{lm}(p,e), h_{rs}(e)] \right\} , \tag{C.3}$$

where

$$\pi^{ij} \equiv \frac{1}{4}\left(p^{ia}e^j_a + p^{ja}e^i_a \right) \tag{C.4}$$

is the new set of canonical momenta, with p^{ia} the canonical momentum conjugate to e_{jb}. However, (C.1) is *not* yet in a suitable form. Since $e^i{}_a$ are the spatial components of the tetrad, these are still, together with $e^i{}_0$, present in the constraints $\mathcal{H}^\mu[\pi^{lm}(p,e), h_{rs}(e)]$ *and* the Lagrange multipliers $\mathcal{N}_\mu(e_{\nu a})$. Moreover, (C.4) is not invertible, since we cannot express $e^a{}_\mu$ as a unique function of the metric. In other words, in (C.3) there are *twelve* pairs of (not yet fully) canonical variables (p^{ia}, e_{jb}), as well as the appropriate Lagrange multipliers $e^a{}_0$.

Hence, we need to include six *additional* constraints in the action (C.3), in order to make the field equations equivalent to Einstein's. It so happens that, from the SO(1,3) transformations of $p\dot{e}$, viz.,

$$p^{ia} \longrightarrow p^{ib}\omega^a_b , \qquad e_{ia} \longrightarrow e_{ib}\omega^b_a , \tag{C.5}$$

we get

$$p^{ia}\dot{e}_{ia} \longrightarrow p^{ia}\dot{e}_{ia} + p^{ib}e_{ic}\omega^a_b\dot{\omega}^c_a , \tag{C.6}$$

for the rotations ω_{ab}, satisfying $\omega^b{}_a\omega^a{}_d = \delta^b{}_a$. This implies that only the skew-symmetric part $\omega^a{}_{[b}\dot{\omega}_{ac]}$ contributes. Then invariance for (C.3) is achieved if (the Lorentz!) constraint

$$\mathcal{J}^{[ab]} \equiv p^{i[a}e^{b]}_i = 0 \tag{C.7}$$

is imposed.

Hence, adding the constraint term $\mathcal{M}_{ab}p^{ia}e^b{}_i$ to the action with the Lagrange multiplier $\mathcal{M}_{ab} = -\mathcal{M}_{ba}$, brings about an extended algebra generated by $\mathcal{J}^{[ab]}$ and $\mathcal{H}^\mu$, which is fully consistent with regard to the metric and tetrad representation. In addition, the constraints $\mathcal{H}^\mu = 0$ are maintained as long as $\mathcal{J}^{[ab]} = 0$ is present. Then the usual Einstein equations are retrieved.

In brief, and touching here upon the ingredients presented in Chap. 4 (see also Chap. 3 and Appendix A), the basic variables in a gravity (Einstein) theory with torsion will be the tetrad $e^a{}_\mu$, its conjugate p^μ_a, and the torsion $\xi_{\mu\nu\lambda}$ with conjugate $P^{\mu\nu\lambda}$. The tetrad (or vierbein) can be understood in terms of an orthonormal basis

e_a and a *coordinate* basis $\mathbf{e}_\mu$, related by $\mathbf{e}_\mu = e^a{}_\mu \mathsf{e}_a$ or $\mathsf{e}_a = e^\mu{}_a \mathbf{e}_\mu$, determining the relations $e^\mu{}_a e_{\mu b} = \eta_{ab}$, $e^a{}_\mu e_{\nu a} = g_{\mu\nu}$. In addition (see (2.8)),

$$\mathcal{N}_i = e_{0a} e^a{}_i \ , \tag{C.8}$$

$$\mathcal{N}^2 = \mathcal{N}_j \mathcal{N}^j - e_{0a} e^a{}_0 \ , \tag{C.9}$$

$$e^0{}_a = -\frac{n_a}{\mathcal{N}} \ , \tag{C.10}$$

$$e_{0a} = \mathcal{N} n_a + \mathcal{N}^i e_{ia} \ . \tag{C.11}$$

Concerning the tetrad, we have for the case of Einstein gravity *with torsion*[2] [2],

$$p_a^0 = 0 \ , \tag{C.12}$$

$$p_a^i = -h^{1/2} \left[K^{(ik)} - \tau^{(ik)} e_{ka} - e^i{}_a (K - \tau) \right] \ , \tag{C.13}$$

from which the (Lorentz) constraint

$$\mathcal{J}^{ab} \equiv p^{ka} e^b{}_k - p^{kb} e^a{}_k = 0 \tag{C.14}$$

is now obtained. The complete Hamiltonian is then (see Chaps. 3 and 4, and Appendix A)

$$H = \mathcal{N}\mathcal{H}_\perp + \mathcal{N}^i \mathcal{H}_i + \mathcal{M}_{ab} \mathcal{J}^{ab} \ . \tag{C.15}$$

Finally, notice that we will have to deal with the torsion $\xi_{\mu\nu\lambda}$ and its conjugate momentum $P^{\mu\nu\lambda}$ regarding (C.15), where it is not present. But see Chap. 4 for an adequate setting in which to discuss this.

2.2 FRW and Bianchi IX from the ADM and Dirac Perspectives

It is still difficult (as it was in the early 1970s) to determine whether (and if so, in what way) the ADM and Dirac formulations differ with regard to their quantum mechanical information content, i.e., properties of the physical system (the universe!) retrieved from the quantum states [3]. Within the ADM method [4], where all the constraints are in principle solved *before* quantization, an *un*ambiguous factor ordering is provided for homogeneous cosmologies. In computational terms, one *major difference* is therefore that the differential equations determining Ψ have *different factor orderings* in the kinetic sector. In fact, for a closed FRW model with dust, the Wheeler–DeWitt equation is

$$\frac{1}{48\pi^2} a^{-1/4} \frac{\mathrm{d}}{\mathrm{d}a} \left\{ a^{-1/2} \left[\frac{\mathrm{d}}{\mathrm{d}a} \left(a^{-1/4} \Psi \right) \right] \right\} - 12\pi^2 a \Psi + N m \Psi = 0 \ , \tag{C.16}$$

[2] Notice that

$$\pi^{km} \equiv \frac{1}{2} \left(p^{ka} e^m{}_a + p^{ma} e^k{}_a \right)$$

will act as canonical momenta to h_{km}.

where N is the total number of dust particles and m their averaged mass, whereas if we convert into Ω time (see Sects. 2.3 and 2.4), where $a = a_0 e^{-\Omega}$, we find an ADM-like form (see (2.54))

$$\frac{\partial^2 \Psi}{\partial \Omega^2} + \frac{\partial \Psi}{\partial \Omega} - \left(60\pi^4 a_0^4 e^{-4\Omega} - 48\pi^2 N M a_0^3 e^{-3\Omega} - \frac{7}{16}\right)\Psi = 0 . \qquad \text{(C.17)}$$

The essential differences with respect to (2.54) are the $\partial\Psi/\partial\Omega$ term and the constant $7/16$ (corresponding to $m^2 = 7/16$ for a specific procedure in the ADM analysis [4]). The term $\partial\Psi/\partial\Omega$ corresponds to a particular factor ordering, and we can in fact obtain it from (C.17), within the ADM formulation.

Another difference between the ADM and the Dirac–Wheeler–DeWitt[3] (DWD) approaches, is in the choice of boundary conditions. This is discussed thoroughly in Sect. 2.7, but note that one early choice within DWD was to take (the so-called infinite wall proposal was advanced somewhat earlier by B. DeWitt (see, e.g., [5, 4])) $\Psi = 0$ at $a = 0$ ($\Omega = -\infty$) when a classically forbidden region is present there. In the ADM approach, there seems no reason to impose $\Psi = 0$ at $\Omega = -\infty$, letting Ψ behave as a free-running particle.

For the Bianchi IX (symmetric, *non*-diagonal) case, the quantum mechanical equation in the DWD scheme is of the form $\Psi^{;X}{}_{;X} = 0$, corresponding to the covariant d'Alembert equation *in the minisuperspace* (see (2.86)) with metric

$$ds^2 = -d\Omega^2 + d\beta_+^2 + d\beta_-^2 + \sinh^2(2\sqrt{3}\beta_-)d\varphi^2 , \qquad \text{(C.18)}$$

leading to

$$\begin{aligned} 0 = &\frac{\partial^2\Psi}{\partial\Omega^2} + \frac{\partial^2\Psi}{\partial\beta_+^2} - (\sinh 2\sqrt{3}\beta_-)^{-1}\frac{\partial}{\partial\beta_-}(\sinh 2\sqrt{3}\beta_-)\frac{\partial\Psi}{\partial\beta_-} \\ &+3(\sinh 2\sqrt{3}\beta_-)^{-2}\frac{\partial^2\Psi}{\partial\varphi^2} + \Big\{e^{-4\Omega}\left[V(\beta_+,\beta_-) - 1\right] \\ &+\mu e^{-3\Omega}\left[1 + \left(2C^2\right)e^{2\Omega+4\beta_+}\right]^{1/2}\Big\}\Psi , \end{aligned} \qquad \text{(C.19)}$$

which separates to $\Psi = \Upsilon(\varphi)\,\chi(\beta_\pm,\Omega)$, with

$$\frac{\partial^2\Upsilon}{\partial\varphi^2} - \varepsilon\Upsilon = 0 \implies \Upsilon \sim e^{\sqrt{\varepsilon}\varphi} , \quad \varepsilon \in \mathbb{R} . \qquad \text{(C.20)}$$

However, we now have to deal with the constraint

$$\mathsf{p}_\varphi\Psi = \mu C\Psi \implies -i\frac{d\Upsilon}{d\varphi} = \mu C\Upsilon , \qquad \text{(C.21)}$$

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[3] Usually referred to as Dirac's formulation, but given the application to quantum cosmology, let us extend to this broader designation.

which leads to $\sqrt{\varepsilon} = \mathrm{i}\mu C$ imaginary! As $\varphi = 0$ and $\varphi = \pi/2$ should represent the same metric, that is satisfactory, but it would also lead to

$$1 = \mathrm{e}^{\mathrm{i}\mu C\pi/2} \implies \mu C = 4n \ , \quad n \in \mathbb{N} \ , \tag{C.22}$$

taking μC to be *initially* arbitrary. This all adds to the differences in the physical implications between the ADM and DWD approaches.

2.3 Momentum Constraints and Spatial Invariance

The essential step [6] is to take an infinitesimal coordinate transformation for the spatial coordinates on each 3-hypersurface. Let us take

$$t \longrightarrow t \ , \qquad x^i \longrightarrow x^i + \delta^i(x^k) \ , \tag{C.23}$$

where $\delta^i(x^k)$ denotes an infinitesimal variation. Consequently, the 3-metric changes according to

$$h_{ij} \longrightarrow h_{ij} + 2\,{}^{(3)}\overline{D}_{(i}\delta_{j)} \ , \tag{C.24}$$

and the 4-metric according to

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + 2\,{}^{(4)}\overline{D}_{(\mu}\delta_{\nu)} \ , \tag{C.25}$$

with $\delta_0 = 0$, ${}^{(3)}\overline{D}$, and ${}^{(4)}\overline{D}$ denoting *here* the covariant derivatives in the 3- and 4-geometry, respectively (see Appendix A and Chap. 2). We then have

$$\Psi\left[h_{ij} + 2\,{}^{(3)}\overline{D}_{(i}\delta_{j)}\right] = \Psi\left[h_{ij}\right] + 2\int \mathrm{d}^3x\ {}^{(3)}\overline{D}_{(i}\delta_{j)}\frac{\delta\Psi}{\delta h_{ij}} \ . \tag{C.26}$$

Integrating the last term by parts and dropping the boundary term (in the compact 3-geometry case), we obtain

$$\delta\Psi \simeq -\int \mathrm{d}^3x\delta_j\,{}^{(3)}\overline{D}_i\left(\frac{\delta\Psi}{\delta h_{ij}}\right) \simeq \int \mathrm{d}^3x\delta_j\mathcal{H}^i\Psi \ , \tag{C.27}$$

whence the wave functions satisfying the momentum constraints are unchanged. This represents the invariance of the theory under 3D diffeomorphisms.

2.4 Geodesics from Quantum Cosmology

In the minisuperspace perspective, an interesting feature can be retrieved, namely, the way a geodesic equation can be assigned (as an interpretation) to the meaning of Einstein's equations within *mini*superspace.

Let us consider a **D**-dimensional minisuperspace with coordinates $\{q^X\}$, X labelling the independent components of the induced 3-metric h_{ij}, taking in addition $\mathcal{N}^i = 0$. The action becomes

$$S \sim \int \mathrm{d}t\left[\frac{1}{2\mathcal{N}}\mathcal{G}_{XY}(q)\dot{q}^X\dot{q}^Y - \mathcal{N}U(q)\right] \ . \tag{C.28}$$

As the reader will have noticed [6, 7], the action (C.28) represents a point particle moving in a potential U. Variation with respect to q^X yields

$$\frac{1}{\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{q}^X}{\mathcal{N}}\right)+\frac{1}{\mathcal{N}^2}\boldsymbol{\Gamma}^X{}_{YZ}\dot{q}^Y\dot{q}^Z=-\mathcal{G}^{XY}\frac{\partial U}{\partial q^X}\ , \tag{C.29}$$

where $\boldsymbol{\Gamma}^X{}_{YZ}$ are the Christoffel symbols *determined from the minisupermetric* $\mathcal{G}_{XY}$. But (C.29) is a geodesic equation with a force term. It should also be noticed that variation of the action (C.28) with respect to $\mathcal{N}$ yields the Hamiltonian constraint (for the solutions of the geodesic equation (C.29) to satisfy):

$$\frac{1}{2\mathcal{N}^2}\mathcal{G}_{XY}(q)\dot{q}^X\dot{q}^Y+U(q)=0\ . \tag{C.30}$$

Overall, the general solution will therefore have $2\mathbf{D}-1$ independent parameters. One corresponds to a choice of origin for the time parameter. Finally, consistency requires[4] that (C.29) and (C.30) correspond, respectively, to the (ij) and (00) components of the original Einstein equations, with the $(0i)$ equation trivially satisfied.

2.5 Killing Vectors and Constants of Motion in Quantum Cosmology

Let us take the Hamiltonian constraint in the form [4]

$$\pi^i{}_j\pi^j{}_i-\frac{1}{2}\pi^2=h\,{}^{(3)}R\ , \tag{C.31}$$

and define $\mathcal{Z}(h)$ as the Hamilton–Jacobi functional, such that, with the definition $\chi^i{}_j\equiv 2h^{ik}\partial/\partial h^{kj}$, we can write

$$\pi_{ij}\equiv\frac{\partial\mathcal{Z}}{\partial h^{ij}}\iff\pi^i{}_j=\frac{1}{2}\chi^i{}_j\,[\mathcal{Z}]\ ,\quad \pi=\frac{1}{2}\chi^j{}_j\,[\mathcal{Z}]\ . \tag{C.32}$$

With the different choice

$$\mathcal{G}^{ijkl}\equiv\frac{1}{2}\left(h^{ik}h^{jl}+h^{jk}h^{il}-h^{ij}h^{kl}\right)\ ,\quad i,j,\ldots\longmapsto X,Y,\ldots\ ,$$

we can write

$$\mathcal{Z}_{,X}\mathcal{Z}_{,Y}\mathcal{G}^{XY}=h\,{}^{(3)}R\equiv\mathrm{R}\ , \tag{C.33}$$

i.e., the Hamiltonian is

$$H=\frac{1}{2}\left(\pi_X\pi_Y\mathcal{G}^{XY}-\mathrm{R}\right)=0\ .$$

The Hamiltonian equations of motion combine to give

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[4] So that substituting a particular ansatz into an action before varying it will yield the same result as substituting the same ansatz into the field equations obtained from variation of the original action [3].

$$\frac{d^2 h^X}{d\ell^2} + \boldsymbol{\Gamma}^X{}_{YT} \frac{dh^Y}{d\ell}\frac{dh^T}{d\ell} = \frac{1}{2}\mathrm{R}^{;X} \equiv \upsilon^X{}_{;Y}\upsilon^Y \ , \tag{C.34}$$

where $\upsilon^X \equiv dh^X/d\ell$ and the semi-colon specifically denotes a covariant derivative in minisuperspace. Equation (C.34) describes a geodesic with a forcing term $\mathrm{R}^{;X}/2$, and $H = 0$ implies that $\upsilon^X \upsilon_X = \mathrm{R}$, hence the connection between the Einstein equations of motion and geodesics in minisuperspace. Now take v as any of the Killing vectors $\chi^i{}_j$ and define the quantity

$$c \equiv -\mathsf{v} \cdot \upsilon \ . \tag{C.35}$$

By differentiation we get

$$\frac{dc}{d\ell} = c_{,X}\upsilon^X = -\left(\mathsf{v}_Y \upsilon^Y\right)_{,X} \upsilon^X = -\frac{1}{2}\mathrm{R}_{;X}\mathsf{v}^X \ . \tag{C.36}$$

Hence, c is a constant if the derivative of R along the direction of v vanishes, i.e., R is invariant under the transformations generated by v.

2.6 Minisuperspace Metrics for FRW and Bianchi IX

Following the description in [4], an invariant (and equivalent) DeWitt metric (see (2.25)) $\mathcal{G}^{XY}(x) \equiv \mathcal{G}^{(ij)(kl)}(x)$ can be (re)defined by requiring that

$$\mathcal{G}^{(ij)(kl)}\mathcal{G}_{(kl)(mn)} = \frac{1}{2}\left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m\right) \ ,$$

leading to

$$\mathcal{G}^{(ij)(kl)} \equiv \frac{1}{2}\sqrt{h}\left(h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}\right) . \tag{C.37}$$

This specific formulation was used by Misner [8, 9] to establish a distance concept between points in superspace, thus related by a quadratic term ds^2, with

$$ds^2 = \mathcal{G}^{(ij)(kl)} dh_{ij} dh_{kl} \ . \tag{C.38}$$

For homogeneous cosmologies, i.e., minisuperspaces, we get an invariant Laplace–Beltrami operator by adding a term proportional to the superspace curvature to $\Psi^{;X}{}_{;X}$, making it conformally invariant (in a minisuperspace manner of speaking). Then using the Misner–Ryan parametrization for Bianchi models in the form

$$h_{ij} dx^i dx^j \approx e^{-2\Omega}\left(e^{2\beta}\right)_{ij}\omega^i \omega^j \ ,$$

we have

$$\mathrm{d}s^2 = \mathrm{e}^{-4\Omega}\left[\left(\mathrm{e}^{2\beta}\right)_{ik}\left(\mathrm{e}^{2\beta}\right)_{jl} + \left(\mathrm{e}^{2\beta}\right)_{jk}\left(\mathrm{e}^{2\beta}\right)_{il} - 2\left(\mathrm{e}^{2\beta}\right)_{ij}\left(\mathrm{e}^{2\beta}\right)_{kl}\right]$$
$$\times\mathrm{d}\left[\mathrm{e}^{2\Omega}\left(\mathrm{e}^{-2\beta}\right)_{ij}\right]\mathrm{d}\left[\mathrm{e}^{2\Omega}\left(\mathrm{e}^{-2\beta}\right)_{kl}\right] , \tag{C.39}$$

from which the following expressions are obtained:

$$\mathrm{d}s^2 \sim -\mathrm{d}\Omega^2 + \mathrm{d}\beta_+^2 + \mathrm{d}\beta_-^2 + \sinh^2(2\sqrt{3}\beta_-)\mathrm{d}\varphi^2 , \tag{C.40}$$
$$\mathrm{d}s^2 \sim -\mathrm{d}\Omega^2 + \mathrm{d}\beta_+^2 + \mathrm{d}\beta_-^2 , \tag{C.41}$$

for the diagonal and symmetric, non-diagonal Bianchi IX cases, respectively.

2.7 WKB and Classical Limit

This is based on Sect. 2.8.1. In order to carry out a WKB expansion [6, 10, 7], and having restored $\hbar$ in the minisuperspace Wheeler–DeWitt equation (2.86), for each Ψ_n, consider the equation

$$\mathcal{H}\Psi = \left[-\frac{1}{2}\hbar^2\nabla^2 + U(q)\right]\mathcal{A}_n \mathrm{e}^{-\mathcal{F}_n/\hbar} . \tag{C.42}$$

The second step is to expand in powers of $\hbar$, in particular, getting the terms associated with $\mathcal{O}(\hbar^0)$ and $\mathcal{O}(\hbar)$:

$$\mathcal{O}(\hbar^0) \longrightarrow -\frac{1}{2}(\nabla\mathcal{F}_n)^2 + U , \tag{C.43}$$
$$\mathcal{O}(\hbar^1) \longrightarrow \nabla\mathcal{F}_n \cdot \nabla\mathcal{A}_n + \frac{1}{2}\mathcal{A}_n\nabla^2\mathcal{F}_n . \tag{C.44}$$

Then decompose $\mathcal{F}_n$ into real and imaginary parts $\mathcal{F}_n \equiv R_n - \mathrm{i}S_n$. From the $\mathcal{O}(\hbar^0)$ term, we get the equations

$$-\frac{1}{2}(\nabla R_n)^2 + \frac{1}{2}(\nabla S_n)^2 + U = 0 , \tag{C.45}$$
$$\nabla R_n \cdot \nabla S_n = 0 , \tag{C.46}$$

where the dot indicates contraction with the minisupermetric $\mathcal{G}_{XY}$. Take the imaginary part to vary much more rapidly than the real part, i.e., $(\nabla R_n)^2 \ll (\nabla S_n)^2$. Then (C.45) is the *Lorentzian* Hamilton–Jacobi equation for S_n, i.e.,

$$\frac{1}{2}\mathcal{G}^{XY}\frac{\partial S_n}{\partial q^X}\frac{\partial S_n}{\partial q^Y} + U(q) = 0 . \tag{C.47}$$

It induces the identification (see (2.85))

$$\pi_X \equiv \frac{\partial S_n}{\partial q^X} . \tag{C.48}$$

But let us further differentiate (C.47) to obtain

$$\frac{1}{2}\mathcal{G}^{XY}{}_{,Z}\frac{\partial S_n}{\partial q^X}\frac{\partial S_n}{\partial q^Y}+\mathcal{G}^{XY}\frac{\partial S_n}{\partial q^X}\frac{\partial^2 S_n}{\partial q^Y\partial q^Z}+\frac{\partial U}{\partial q^Z}=0\,. \tag{C.49}$$

Finally, *define* the minisuperspace vector field

$$\frac{\mathrm{d}}{\mathsf{ds}}\equiv\mathcal{G}^{XY}\frac{\partial S_n}{\partial q^X}\frac{\partial}{\partial q^Y}\,, \tag{C.50}$$

and use (C.48) and (C.49) to obtain

$$\frac{\mathrm{d}\pi_Z}{\mathsf{ds}}+\frac{1}{2}\mathcal{G}^{XY}{}_{,Z}\pi_X\pi_Y+\frac{\partial U}{\partial q^Z}=0\,, \tag{C.51}$$

which, in view of (2.83), is just the geodesic equation (C.30), with s as 'proper time'.

Now we proceed to order $\mathcal{O}(\hbar)$. With $(\nabla R_n)^2 \ll (\nabla S_n)^2$, the terms involving R_n can be neglected, yielding

$$\mathcal{G}^{XY}\frac{\partial S_n}{\partial q^X}\frac{\partial \mathcal{A}_n}{\partial q^Y}\equiv\frac{\mathrm{d}\mathcal{A}_n}{\mathsf{ds}}=-\frac{1}{2}\mathcal{A}_n\nabla^2 S_n\,, \tag{C.52}$$

and hence implying the first-order WKB wave function

$$\Psi_n=C_n\exp\left(\mathrm{i}S_n-\frac{1}{2}\int\mathsf{ds}\,\nabla^2 S_n\right)\,, \tag{C.53}$$

where C_n is an arbitrary (complex) constant ($\hbar=1$).

Problems of Chap. 3

3.1 SUSY Generator as Spin 1/2 Fermion

Since fermions and bosons behave differently under rotations, this implies that the generators of SUSY will *not* be invariant under spatial rotations. Consider therefore a unitary operator $\mathcal{U}$ representing a rotation of 360° around some axis. Then from (3.1), using $\mathcal{S}$ for the SUSY charges[5] (possible generators or constraints)

[5] In Chap. 3, we used Q_A to represent the SUSY charges (or generators) associated with *global* transformations. When proceeding to *local* SUSY, we will use $\mathcal{S}_A$ instead. That will be the situation from Chap. 4 on, so at this point and throughout Exercises 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6, we present the expressions with $\mathcal{S}_A$, in order to provide a foothold for the reader.

$$\mathcal{U}\mathcal{S}\mathcal{U}^{-1}\mathcal{U}\,|\text{fermion}\rangle = \mathcal{U}\mathcal{S}\,|\text{fermion}\rangle = \mathcal{U}\,|\text{boson}\rangle = |\text{boson}\rangle\ ,$$
$$\mathcal{U}\mathcal{S}\mathcal{U}^{-1}\mathcal{U}\,|\text{boson}\rangle = \mathcal{U}\mathcal{S}\,|\text{boson}\rangle = \mathcal{U}\,|\text{fermion}\rangle = -\,|\text{fermion}\rangle\ .$$

we get that $\mathcal{U}\mathcal{S}\mathcal{U}^{-1} = -\mathcal{S}$, i.e., SUSY generators behave like spin 1/2 fermions under rotations.

3.2 Retrieving the Algebra (3.9), (3.10), and (3.11)

The form of the SUSY algebra can be extracted as follows [11–15]:

- An infinitesimal transformation for a scalar field ϕ can be written in the form $\delta\phi = \mathrm{i}\left(\varepsilon^A \mathcal{S}_A\right)\varphi$, with

$$\varepsilon^A = \begin{pmatrix} \varepsilon^A \\ \overline{\varepsilon}_{B'} \end{pmatrix}\ , \qquad \mathcal{S}_A = \begin{pmatrix} \mathcal{S}_A \\ \overline{\mathcal{S}}^{B'} \end{pmatrix}\ ,$$

  as Grassmann quantitites and Majorana spinors, and their components as Weyl spinors.
- Define $\delta_1 \equiv \mathrm{i}\left(\varepsilon_1^A \mathcal{S}_A + \overline{\varepsilon}_{1A'}\overline{\mathcal{S}}^{A'}\right)$, and compute $[\delta_1, \delta_2]$ using $\{\varepsilon, \mathcal{S}\} = 0$ and $\varepsilon_1^A \mathcal{S}_A = -\varepsilon_{1A}\mathcal{S}^A$.
- Then, from the known expressions for a *generic* field, i.e.,

$$[\delta_1, \delta_2]\,\varphi = 2\overline{\varepsilon}_2\gamma^\rho\varepsilon_1\mathcal{P}_\rho\varphi\ ,$$

  followed by taking $\mathcal{P}_\rho \equiv -\mathrm{i}\partial_\rho$, and using

$$\varepsilon^\dagger = \left(\overline{\varepsilon}^{A'}, \varepsilon_A\right) \implies \overline{\varepsilon} = \left(\varepsilon_A, \overline{\varepsilon}^{A'}\right)\ ,$$

  for $\overline{\varepsilon}_2\gamma^\rho\varepsilon_1$, this gives $-\varepsilon_2^A\sigma_A^{\rho B'}\overline{\varepsilon}_{1B'} + \eta_1^A\sigma_A^{\rho B'}\overline{\varepsilon}_{2B'}$.
- Then $[\delta_1, \delta_2] = -2\eta_2^A\sigma_A^{\rho B'}\mathcal{P}_\rho\overline{\eta}_{1B'} + 2\eta_1^A\sigma_A^{\rho B'}\mathcal{P}_\rho\overline{\varepsilon}_{2B'}$.
- From the computations for $[\delta_1, \delta_2]$, the known expressions follow:

$$\left\{\mathcal{S}_A, \overline{\mathcal{S}}_{A'}\right\} = 2\sigma^\mu_{aA'}\mathcal{P}_\mu\ , \tag{C.54}$$
$$\left\{\mathcal{S}_A, \mathcal{S}_B\right\} = 0\ , \tag{C.55}$$
$$\left\{\overline{\mathcal{S}}_{A'}, \overline{\mathcal{S}}_{B'}\right\} = 0\ . \tag{C.56}$$

With (A.5) and (3.2), (3.3), (3.4), and (3.5), these constitute the (SUSY) Poincaré (graded) Lie algebra.

But the attentive reader may be asking how we can justify the use of $[\delta_1, \delta_2]\,\phi = 2\overline{\varepsilon}_2\gamma^\rho\varepsilon_1\mathcal{P}_\rho\varphi$. In fact, it all comes down to establishing how the different fields will have to vary under SUSY in order to produce SUSY invariant actions (see Sect. 3.2.3).

3.3 Spin Raising

Taking $\mathcal{L}_{12} \equiv \mathcal{J}_3$ and using $\overline{\mathcal{S}}^{0'} = -(\mathcal{S}_1)^\dagger$ and $\overline{\mathcal{S}}^{1'} = (\mathcal{S}_0)^\dagger$, we have [12]

$$[\mathcal{J}_3, \mathcal{S}_0] = \frac{1}{2}\mathcal{S}_0 \ , \qquad [\mathcal{J}_3, \mathcal{S}_1] = -\frac{1}{2}\mathcal{S}_1 \ , \tag{C.57}$$

$$[\mathcal{J}_3, \overline{\mathcal{S}}^{0'}] = \frac{1}{2}\overline{\mathcal{S}}^{0'} \ , \qquad [\mathcal{J}_3, \overline{\mathcal{S}}^{1'}] = -\frac{1}{2}\overline{\mathcal{S}}^{1'} \ . \tag{C.58}$$

Hence, $\mathcal{S}_0$ and $\overline{\mathcal{S}}^{0'}$ raise (the z-component of) the spin (e.g., helicity) by half a unit, while $\mathcal{S}_1$ and $\overline{\mathcal{S}}^{1'}$ lower it by half a unit. This is particularly important when establishing particle (or state) multiplets under SUSY, and also when constructing invariants, such as a Lagrangian.

3.4 Number of Bosonic and Fermionic Degrees of Freedom

- Let the fermion number be $\mathcal{F}$ such that $\mathcal{F} = 1$ on a fermionic state and $\mathcal{F} = 0$ on a bosonic state, i.e., $(-)^{\mathcal{F}}$ is $+1$ on bosons and -1 on fermions.
- From the fact that $\mathcal{S}_A$ changes the fermion number by one unit (see Exercise 3.3), $(-1)^{\mathcal{F}}\mathcal{S}_A = -\mathcal{S}_A(-1)^{\mathcal{F}}$, so $(-)^{\mathcal{F}}$ *anticommutes* with $\mathcal{S}$.
- We calculate the trace (over all possible states) of the energy operator weighted by $(-)^{\mathcal{F}}$:

$$\begin{aligned}
2\sigma^\mu_{AB'}\mathrm{Tr}\left[(-)^{\mathcal{F}}\mathcal{P}_\mu\right] &= 2\sum_i \langle i|\,(-)^{\mathcal{F}}\sigma^\mu_{AB'}\mathcal{P}_\mu\,|i\rangle = \sum_i \langle i|\,(-)^{\mathcal{F}}\{\mathcal{S}_A, \overline{\mathcal{S}}_{B'}\}\,|i\rangle \\
&= \sum_i \langle i|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|i\rangle + \sum_i \langle i|\,(-)^{\mathcal{F}}\overline{\mathcal{S}}_{B'}\mathcal{S}_A\,|i\rangle \\
&= \sum_i \langle i|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|i\rangle + \sum_{i,j} \langle i|\,(-)^{\mathcal{F}}\overline{\mathcal{S}}_{B'}\,|j\rangle\,\langle j|\,\mathcal{S}_A\,|i\rangle \\
&= \sum_i \langle i|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|i\rangle + \sum_{i,j} \langle j|\,\mathcal{S}_A\,|i\rangle\,\langle i|\,(-)^{\mathcal{F}}\overline{\mathcal{S}}_{B'}\,|j\rangle \\
&= \sum_i \langle i|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|i\rangle + \sum_j \langle j|\,\mathcal{S}_A(-)^{\mathcal{F}}\overline{\mathcal{S}}_{B'}\,|j\rangle \\
&= \sum_i \langle i|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|i\rangle - \sum_j \langle j|\,(-)^{\mathcal{F}}\mathcal{S}_A\overline{\mathcal{S}}_{B'}\,|j\rangle = 0 \ .
\end{aligned} \tag{C.59}$$

Then, using the cyclicity of the trace and assuming a non-vanishing momentum $\mathcal{P}_\mu$, we get

$$\mathrm{Tr}\,(-)^{\mathcal{F}} = 0 \ , \tag{C.60}$$

if the trace is taken over any finite-dimensional representation.

3.5 Massless and Massive Supermultiplets with N=1 or N=2 and with or without Central Charges

Massless Case without Central Charges. For the case of extended SUSY and assuming that all central charges Z^{IJ} vanish (so the internal group is U(N)), all $\mathcal{S}^I_A$ and $\overline{\mathcal{S}}^I_{A'}$ anticommute among themselves. In the reference frame, where $\mathcal{P}_\mu = (E, 0, 0, E)$, we have

$$\sigma^\mu \mathcal{P}_\mu = \begin{bmatrix} 0 & 0 \\ 0 & -2E \end{bmatrix} \Longrightarrow \{\mathcal{S}^I_A , \overline{\mathcal{S}}^J_{A'}\} = \begin{bmatrix} 0 & 0 \\ 0 & -4E \end{bmatrix}_{AA'} \delta^{IJ} . \tag{C.61}$$

But this means $\{\mathcal{S}^I_1, \overline{\mathcal{S}}^J_{1'}\} = 0$, implying that $\mathcal{S}^I_1 = \overline{\mathcal{S}}^J_{1'} = 0$, $\forall I, J$. For the remaining $\mathcal{S}^I_0, \overline{\mathcal{S}}^J_{0'}$, take them as anticommuting annihilation and creation operators:

$$a_I \equiv \frac{1}{\sqrt{4E}} \mathcal{S}^I_0 , \quad a^\dagger_J \equiv \frac{1}{\sqrt{4E}} \overline{\mathcal{S}}^J_{0'} \Longrightarrow \left\{a_I, a^\dagger_J\right\} = \delta_{IJ} , \tag{C.62}$$

$$\{a_I, a_J\} = \left\{a^\dagger_I, a^\dagger_J\right\} = 0 . \tag{C.63}$$

Then, the vacuum state $|\mathrm{O}\rangle$ is annihilated by all the a_I, $\mathcal{S}^I_0$ lowers the helicity by one half, and $\overline{\mathcal{S}}^J_{0'}$ raises it by one half, so the supermultiplet will be formed by

$$|\mathrm{O}\rangle , \qquad a^\dagger_I |\mathrm{O}\rangle \equiv |\mathrm{O} + 1/2\rangle_I , \tag{C.64}$$

$$a^\dagger_I a^\dagger_J |\mathrm{O}\rangle = |\mathrm{O} + 1\rangle_{IJ} , \quad \ldots , \quad a^\dagger_1 a^\dagger_2 \ldots a^\dagger_N |\mathrm{O}\rangle = |\mathrm{O} + N/2\rangle , \tag{C.65}$$

with $\mathrm{O} + N/2$ taking values in 0, 1/2, 1, ... For example, when $N = 2$, a supermultiplet contains more specifically (note the degeneracy)

$$\left(\mathrm{O}, \mathrm{O} + 1/2, \mathrm{O} + 1/2, \mathrm{O} + 1\right) ,$$

i.e., a vector multiplet with (0, 1/2, 1/2, 1) and its CPT conjugate corresponding to a vector (gauge boson), two Weyl fermions, and a complex scalar, again all in the adjoint representation of the gauge group [11, 12, 16].

Massive Case with Central Charges. With central charges Z^{IJ} and in the rest frame $\mathcal{P}_\mu = (m, 0, 0, 0)$, the SUSY algebra (3.9), (3.10), and (3.11) becomes

$$\left\{\mathcal{S}^I_A, (\mathcal{S}^J_B)^\dagger\right\} = 2m\delta_{AB}\delta^{IJ} , \tag{C.66}$$

$$\left\{\mathcal{S}^I_A, \mathcal{S}^J_B\right\} = \varepsilon_{AB} Z^{IJ} , \tag{C.67}$$

$$\left\{(\mathcal{S}^I_A)^\dagger, (\mathcal{S}^J_B)^\dagger\right\} = \varepsilon_{AB} (Z^{IJ})^* , \tag{C.68}$$

while the antisymmetric matrix of central charges becomes[6]

$$Z^{IJ} \equiv \begin{bmatrix} 0 & q_1 & 0 & 0 & \\ -q_1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & q_2 & \cdots \\ 0 & 0 & -q_2 & 0 & \\ 0 & 0 & \vdots & & \end{bmatrix} , \tag{C.69}$$

with all $q_n \geq 0, n = 1, \dots N/2$. Now define

$$a^1_A = \frac{1}{\sqrt{2}} \left[\mathcal{S}^1_A + \varepsilon_{AB} \left(\mathcal{S}^2_B \right)^\dagger \right] , \tag{C.70}$$

$$b^1_A = \frac{1}{\sqrt{2}} \left[\mathcal{S}^1_A - \varepsilon_{AB} \left(\mathcal{S}^2_B \right)^\dagger \right] , \tag{C.71}$$

$$a^2_A = \frac{1}{\sqrt{2}} \left[\mathcal{S}^3_A + \varepsilon_{AB} \left(\mathcal{S}^4_B \right)^\dagger \right] , \tag{C.72}$$

$$b^2_A = \frac{1}{\sqrt{2}} \left[\mathcal{S}^3_A - \varepsilon_{AB} \left(\mathcal{S}^4_B \right)^\dagger \right] , \tag{C.73}$$

and continuing for other values of N, yielding the algebra

$$\left\{ a^r_A, (a^s_B)^\dagger \right\} = (2m - q_r)\delta_{rs}\delta_{AB} , \tag{C.74}$$

$$\left\{ b^r_A, (b^s_B)^\dagger \right\} = (2m + q_r)\delta_{rs}\delta_{AB} , \tag{C.75}$$

$$\left\{ a^r_A, (b^s_B)^\dagger \right\} = \{ a^r_A, a^s_B \} = \dots = 0 , \tag{C.76}$$

where m and q are the mass and central charge of the supermultiplet. For a state $|m, q\rangle$ with positive norm, from (C.74), positivity of the Hilbert space requires that

$$\langle m, q| \, a^r_A (a^r_A)^\dagger \, |m, q\rangle + \langle m, q| \, (a^r_A)^\dagger a^r_A \, |m, q\rangle = 2m - q_r \implies 2m \geq |q_r| , \tag{C.77}$$

for all r. If some of the q_r *saturate* the bound,[7] i.e., are equal to $2m$, then the corresponding operators set states with zero norm and the state is annihilated by half of the supercharges, i.e., invariant under half of the SUSY algebra.

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[6] By an appropriate U(N) rotation among the $\mathcal{S}^I$, where N is even, since otherwise there is an extra zero eigenvalue of the matrix Z which can be handled trivially.

[7] In the massless case, the bound becomes $0 \geq |q_n|$ and thus we always have $q_n = 0$. There cannot be central charges. The bound is always saturated. Only exactly half of the fermionic generators survive.

3.6 SUSY Invariant Lagrangians in (SUSY) Superspace and the Wess–Zumino Model

Any Lagrangian of the form

$$\int d^2\theta d^2\overline{\theta}\, \mathsf{F}(x,\theta,\overline{\theta}) + \int d^2\theta\, \mathsf{W}(\Phi) + \int d^2\overline{\theta}\, [\mathsf{W}(\Phi)]^\dagger \tag{C.78}$$

is automatically SUSY invariant. It transforms at most by a total derivative in spacetime.

The supersymmetry of the action resulting from the spacetime integral of the Lagrangian (C.78) is as follows. The SUSY variation of any superfield is given by (3.52). Because ε and $\overline{\varepsilon}$ are constant spinors, the $\mathcal{S}$ and $\overline{\mathcal{S}}$ are differential operators in superspace and (3.52) is a total derivative [16]:

$$\delta\mathsf{F} = \frac{\partial}{\partial\theta^A}(-\varepsilon^A\mathsf{F}) + \frac{\partial}{\partial\overline{\theta}^{A'}}(-\overline{\varepsilon}^{A'}\mathsf{F}) + \frac{\partial}{\partial x^\mu}\big[-(\varepsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\varepsilon})\mathsf{F}\big] . \tag{C.79}$$

Carrying out the $\int d^2\theta d^2\overline{\theta}$ integrals leaves only the last term, a total spacetime derivative.

In particular, a chiral or antichiral superfield, Φ or $\overline{\Phi}$, respectively (or $\mathsf{W}(\Phi)$ or $\overline{\mathsf{W}}(\overline{\Phi}) = [\mathsf{W}(\Phi)]^\dagger$, respectively), with variables θ and y or $\overline{\theta}$ and $\overline{y}$, respectively, yields

$$\delta\Phi = \frac{\partial}{\partial\theta^A}\big[-\varepsilon^A\Phi(y,\theta)\big] + \frac{\partial}{\partial y^\mu}\big[-(\varepsilon\sigma^m\overline{\theta} - \theta\sigma^m\overline{\varepsilon})\Phi(y,\theta)\big] . \tag{C.80}$$

Carrying out the $\int d^2\theta$ integral, or the $\int d^2\overline{\theta}$ integral, as appropriate, leaves only the last term, which becomes a total derivative in spacetime.

Let us now apply the above. Consider the discussion of the SUSY invariance of the Wess–Zumino (WZ) Lagrangian, which constitutes an efficient and economical way to produce SUSY invariant actions. The reader may appreciate this through the following recipe [11, 12, 15]:

- Products of (anti) chiral superfields are still (anti) chiral superfields.
- A *superpotential* $\mathsf{W}(\Phi)$ can then be constructed (e.g., chiral), depending on several different Φ_I. Using the y and θ variables, a Taylor expansion produces

$$\mathsf{W}(\Phi) = \mathsf{W}\big(\phi(y)\big) + \sqrt{2}\,\frac{\partial\mathsf{W}}{\partial\phi_\mathsf{I}}\bigg|_{\phi(y)}\theta\psi_\mathsf{I}(y) + \theta\theta\left[\frac{\partial\mathsf{W}}{\partial\phi_\mathsf{I}}\bigg|_{\phi(y)} f_\mathsf{I}(y) - \frac{1}{2}\frac{\partial^2\mathsf{W}}{\partial\phi_\mathsf{I}\partial\phi_\mathsf{J}}\bigg|_{\phi(y)}\psi_\mathsf{I}(y)\psi_\mathsf{J}(y)\right] . \tag{C.81}$$

- The terms $\int d^2\theta\mathsf{W}(\Phi)$ plus Hermitian conjugate in the Lagrangian have the form of a potential.

- The kinetic terms must therefore be provided by the term $\int d^2\theta d^2\overline{\theta}\Phi$:
  - Choose, for simplicity, the *real* term $\Phi \equiv \phi^\dagger\phi$.
  - Expand the y^μ in terms of the x^μ .
  - Retain the terms[8] $\theta\theta\overline{\theta}\,\overline{\theta}$:

$$\Phi^\dagger\Phi\Big|_{\theta\theta\overline{\theta}\,\overline{\theta}} = -\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi + \frac{\mathrm{i}}{2}(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}) + f^\dagger f$$
$$+\text{total derivative}\,. \qquad \text{(C.82)}$$

- The action is then

$$S = \int d^4x \left(|\partial_\mu\phi_{\mathsf{I}}|^2 - \mathrm{i}\psi_{\mathsf{I}}\sigma^\mu\partial_\mu\overline{\psi}_{\mathsf{I}} + f_{\mathsf{I}}^\dagger f_{\mathsf{I}} + \frac{\partial \mathsf{W}}{\partial\phi_{\mathsf{I}}} f_{\mathsf{I}} + \text{h.c.} \right.$$
$$\left. -\frac{1}{2}\frac{\partial^2\mathsf{W}}{\partial\phi_{\mathsf{I}}\partial\phi_{\mathsf{J}}}\psi_{\mathsf{I}}\psi_{\mathsf{J}} + \text{h.c.}\right) \,. \qquad \text{(C.83)}$$

Note the *auxiliary* fields f_{I}. They have *no* kinetic term. They are eliminated through the use of their algebraic equations of motion

$$f_{\mathsf{I}}^\dagger = -\frac{\partial\mathsf{W}}{\partial\phi_{\mathsf{I}}}\,, \qquad \text{(C.84)}$$

in the action, leading to

$$S = \int d^4x \left[-|\partial_\mu\phi_{\mathsf{I}}|^2 - \mathrm{i}\psi_{\mathsf{I}}\sigma^\mu\partial_\mu\overline{\psi}_{\mathsf{I}} - \left|\frac{\partial\mathsf{W}}{\partial\phi_{\mathsf{I}}}\right|^2 - \frac{1}{2}\frac{\partial^2\mathsf{W}}{\partial\phi_{\mathsf{I}}\partial\phi_{\mathsf{J}}}\psi_{\mathsf{I}}\psi_{\mathsf{J}} \right.$$
$$\left. -\frac{1}{2}\left(\frac{\partial^2\mathsf{W}}{\partial\phi_{\mathsf{I}}\partial\phi_{\mathsf{J}}}\right)^\dagger \overline{\psi}_{\mathsf{I}}\overline{\psi}_{\mathsf{J}} \right] \,, \qquad \text{(C.85)}$$

where the scalar potential V is now *determined* in terms of the superpotential W as

$$V = \sum_{\mathsf{I}} \left|\frac{\partial\mathsf{W}}{\partial\phi_{\mathsf{I}}}\right|^2 \,. \qquad \text{(C.86)}$$

---

[8] Usually called the d term [11, 12, 14, 17, 15, 18]. It is due to $\Phi_{\mathsf{I}}^\dagger\Phi_{\mathsf{J}}$ yielding a term whose structure is as found in a vector superfield (see Sect. 3.3.2), and where the $\theta\theta\overline{\theta}\,\overline{\theta}$ term is associated with a scalar field d. The point is that the $\theta\theta$ term in $\Phi_{\mathsf{I}}\Phi_{\mathsf{J}}$ (called the F term due to the presence of $f\theta\theta$ in the expansion of $\Phi(x,\theta,\overline{\theta})$, see (3.62)) and also the d term in $\Phi_{\mathsf{I}}^\dagger\Phi_{\mathsf{J}}$, are *invariant* under SUSY transformations (up to total derivatives). The SUSY-invariant Lagrangian can be presented in the form

$$L = \sum_{\mathsf{I}} \left[\Phi_{\mathsf{I}}^\dagger\Phi_{\mathsf{I}}\right]_{\mathsf{d}\,\text{term}} + \left[\mathsf{W}(\Phi) + \text{h.c.}\right]_{F\,\text{term}}\,,$$

with $\mathsf{W}(\Phi)$ the superpotential.

In the case of a *single* chiral superfield Φ and a cubic superpotential

$$\mathsf{W}(\phi) = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 ,$$

the action becomes[9]

$$S_{\mathrm{WZ}} = \int \mathrm{d}^4x \Big\{ - |\partial_\mu\phi|^2 - \mathrm{i}\psi\sigma^\mu\partial_\mu\overline{\psi} - m^2|\phi|^2 - \frac{m}{2}(\psi\psi + \overline{\psi}\,\overline{\psi}) - mg\Big[\phi^\dagger\phi^2 + (\phi^\dagger)^2\phi\Big] - g^2|\phi|^4 + g(\phi\psi\psi + \phi^\dagger\overline{\psi}\,\overline{\psi})\Big\} . \quad \text{(C.87)}$$

Problems of Chap. 4

4.1 Second Class Constraints and Dirac Brackets in N=1 SUGRA

Summarising and using the notation of Chap. 4 and Appendix A (see also [2]):

- The Dirac bracket (as used there) is defined by

$$[A, B]_{\mathrm{D}} = [A, B]_{\mathrm{P}} - \Big[A, \ell^i_{[a]}\Big]_{\mathrm{P}} D_{ij[a][b]} \Big[\ell^j_{[b]}, B\Big]_{\mathrm{P}} . \quad \text{(C.88)}$$

- We are using (see Appendix A)

$$\ell^i \equiv \pi^i - h^{1/2}\overline{\psi}_j\gamma^\perp\sigma^{ji} , \quad \text{(C.89)}$$

$$D_{ij[a][b]} \equiv \Big[C^{ij}_{[a][b]}\Big]^{-1} . \quad \text{(C.90)}$$

- We calculate

$$\begin{aligned} C^{ij}_{[a][b]} &= \Big[\pi^i_{[a]} - \frac{\mathrm{i}}{2}\varepsilon^{mn0i}\psi_{m[d]}\,(\gamma_0\gamma_5\gamma_n)_{[d][a]}\,, \pi^j_{[b]} - \frac{\mathrm{i}}{2}\varepsilon^{mn0j}\psi_{m[c]}\,(\gamma_0\gamma_5\gamma_n)_{[c][b]}\Big]_{\mathrm{P}} \\ &= -\frac{\mathrm{i}}{2}\varepsilon^{in0j}\psi_{m[d]}\,(\gamma_0\gamma_5\gamma_n)_{[a][b]} - \frac{\mathrm{i}}{2}\varepsilon^{jnoi}\psi_{m[d]}\,(\gamma_0\gamma_5\gamma_n)_{[b][a]} \\ &= -\mathrm{i}\varepsilon^{nji}\,(\gamma_0\gamma_5\gamma_n)_{[a][b]} = 2h^{1/2}\big(\gamma_0\sigma^{ji}\gamma^\perp\big)_{[a][b]} , \end{aligned} \quad \text{(C.91)}$$

$$D_{ij[a][b]} = -\frac{1}{2}h^{-1/2}\,\big(\gamma_\perp\gamma_j\gamma_i\gamma_0\big)_{[a][b]} . \quad \text{(C.92)}$$

---

[9] This is of course the Wess–Zumino action. See Sect. 3.2.3 and (3.32).

- The Dirac brackets are then found, using

$$\left[p^{ia}(x), \lambda^m\right]_{\mathrm{P}} = -\frac{\mathrm{i}}{2}\varepsilon^{ism}\overline{\psi}_s\gamma_5\gamma^a$$

and establishing that π^k can be replaced by $h^{1/2}\overline{\psi}_j\gamma^\perp\sigma^{ji}$, whenever it occurs. The result is

$$\left[\psi_{m[a]}(x), \psi_{n[b]}(x')\right]_{\mathrm{D}} = \frac{1}{2}h^{-1/2}\left(\gamma_\perp\gamma_n\gamma_m\gamma_0\right)_{[a][b]}\delta(x,x') , \tag{C.93}$$

$$\begin{aligned}\left[p^{ia}(x), \psi_n(x')\right]_{\mathrm{D}} &= -\left[p^{ia}(x), \lambda^m\right]_{\mathrm{P}} D_{mn} \\ &= \frac{1}{2}\overline{\psi}_s\gamma^a\gamma_n\sigma^{si}\gamma_0\delta(x,x') ,\end{aligned} \tag{C.94}$$

$$\begin{aligned}\left[p^{ia}(x), p^{jb}(x')\right]_{\mathrm{D}} &= -\left[p^{ia}(x), \lambda^m\right]_{\mathrm{P}} D_{mn}\left[\lambda^n, p^{jb}(x')\right]_{\mathrm{P}} \\ &= \frac{1}{2}h^{1/2}\overline{\psi}_s\gamma^a\sigma^{jr}\gamma_\perp\sigma^{is}\gamma^b\psi_r\delta(x,x') .\end{aligned} \tag{C.95}$$

4.2 Constraints as 'Square-Root' and SUSY

The relation between two theories through a square-root framework (as exposed here) intertwines the inclusion of spin degrees of freedom with the invariance of the overall description under transformations mixing usual (bosonic) coordinates with Grassmanian (fermionic) variables (i.e., pointing toward SUSY).

Consider the case (see also [19–22]) of a relativistic point particle, and write the Dirac equation as

$$\mathcal{S} \equiv \theta_\mu\mathcal{P}^\mu + \theta_5 m \cong 0 ,$$

with

$$\theta_\mu \equiv \mathrm{i}\gamma_5\gamma_\mu/\sqrt{2} , \qquad \theta_5 \equiv \gamma_5/\sqrt{2} ,$$

and write the associated Klein–Gordon equation as

$$H = \mathcal{P}^\mu\mathcal{P}_\mu + m^2 \cong 0 ,$$

related through

$$\{\mathcal{S},\mathcal{S}\} = -H , \qquad [\mathcal{S}, H] = [H, H] = 0 .$$

The Dirac equation is the 'square root' of the Klein–Gordon equation when (within the quantum description) we retrieve the physical states. Notice the mixing generated by $\mathcal{S}$, which includes $\delta x^\mu = \varepsilon\theta^\mu$ and $\delta\theta^\mu = -\mathrm{i}\varepsilon\mathcal{P}^\mu$.

This method can be extended to wider settings, such as the spinless string, where arbitrary deformations of the string are possible (normal and parallel). A square root description as above is possible, and spin degrees of freedom are introduced, thereby inducing SUSY transformations.

For general relativity, the initial object to consider is a 3D surface, with the constraints and variables as described in Chap. 2. One uses the tetrad e^a_μ and corresponding canonical momentum π^μ_a, with constraints $\mathcal{H}_\perp$, $\mathcal{H}_i$, and $\mathcal{J}_{ab}$, associated with normal and tangential deformations of the surface and local rotations, respectively. As $\mathcal{H}_\perp$ is quadratic in the momenta, we can (as in the cases above), associate a square-root structure, whereby at each point of spacetime an (intrinsic) spin degree of freedom is introduced. This leads to $N = 1$ SUGRA (!), which may thus be interpreted as a theory of *spinning* space in the context of (4.51)–(4.65). An equivalent way to present it [2] is to say that introducing spin into general relativity via a square-root procedure for the Hamiltonian leads to *local* SUSY, implying that $N = 1$ SUGRA is *the* physical extension of general relativity. Equivalently, when the square-root of general relativity is extracted in the above framework, the solution is $N = 1$ SUGRA [23–25].

4.3 Expressing $n^{AA'}$ in Terms of $e_i^{AA'}$

Use expressions (A.43), (A.44), (A.45), and (A.46), together with (A.48), (A.49), (A.50), (A.51), (A.52), (A.53), and (A.54), presented in Sect. A.4 of Appendix A.

4.4 Simplifying the Supersymmetry Constraints

Use Appendix A to work out the expression

$$^{(3)}D_j \mathsf{T}^{AA'} = \partial_j \mathsf{T}^{AA'} + {}^{(3)}\omega^A_B \mathsf{T}^{BA'} + {}^{(3)}\overline{\omega}^{A'}_{B'} \mathsf{T}^{AB'} , \tag{C.96}$$

where ${}^{(3)}\omega^A_B$ and ${}^{(3)}\overline{\omega}^{A'}_{B'}$ are the two parts of the spin connection (see (A.77)), using the decomposition

$$^{(3)}\omega_i^{AA'BB'} = {}^{(3)\mathrm{s}}\omega_i^{AA'BB'} + {}^{(3)}\kappa_i^{AA'BB'} . \tag{C.97}$$

Problems of Chap. 5

5.1 Evaluating ${}^3D_i\varepsilon_A$ for $\delta\psi_i^A$

The variation of ψ_i^A for Lorentz, general coordinate, and SUSY transformations, is [26]

$$\delta\psi_i^A = \left(\mathcal{N}^A_B \psi_i^B\right)_{\text{Lorentz}} + \left(\xi^j \partial_j \psi_i^A + \psi_j^A \partial_i \xi^j\right)_{\text{general}} + \left[2\mathsf{k}^{-1}\, {}^{(3)}D_i \varepsilon^A\right]_{\text{SUSY}} . \tag{C.98}$$

We then calculate for ${}^{(3)}D_i\varepsilon^A$ the spatial versions of the torsion-free connection forms ${}^{(3\mathrm{s})}\omega_i^{AB}$ and the contorsion tensor κ_i^{AB} :

$$
{}^{(3s)}\omega_i^{AB} = \left(\frac{\dot{a}}{a\mathcal{N}} + \frac{\mathrm{i}}{a}\right) n^A_{\ A'} e^{BA'}{}_i \ , \tag{C.99}
$$

$$
\kappa_i^{AB} = -\frac{\mathrm{i}\mathsf{k}^2}{4\mathcal{N}}\left(\psi_F \psi^F{}_0 + \overline{\psi}_{F'0}\overline{\psi}^{F'}\right) n^A_{\ A'} e^{BA'}{}_i + \frac{\mathrm{i}\mathsf{k}^2}{4} e_i^{(AE'} \varepsilon^{B)E} \psi_E \overline{\psi}_{E'} \ . \tag{C.100}
$$

Using (5.4) and (5.5) with the correspondence $\xi^i = -\xi^{AA'} e^i_{AA'}$, $\xi^{AA'} = \mathrm{i}\xi^{AB} n_B^{A'}$, $\overline{\xi}^{A'B'} = 2\xi^{AB} n_B^{B'} n_A^{A'}$, we get (5.6).

5.2 Using ψ_0^A or ρ^A

Employing ψ_0^A (see Chap. 5) instead of

$$
\rho^A = \frac{\mathrm{i}(\mathsf{k}\varsigma)^{1/2}}{2\sqrt{6}a^{1/2}} \psi^A{}_0 + \frac{\mathrm{i}\varsigma\mathcal{N}}{12\mathsf{k}a^2} n^{AA'} \overline{\psi}_{A'} \ ,
$$

the supersymmetry and Hamiltonian constraints read (in the pure case):

$$
\mathcal{S}_A = \psi_A \pi_a - 6\mathrm{i}a\psi_A + \frac{\mathrm{i}}{2a} n_A^{\ E'} \psi^E \psi_E \overline{\psi}_{E'} \ , \tag{C.101}
$$

$$
\overline{\mathcal{S}}_{A'} = \overline{\psi}_{A'}\pi_a + 6\mathrm{i}a\overline{\psi}_{A'} - \frac{\mathrm{i}}{2a} n_E^{\ A'} \overline{\psi}^{E'} \psi_E \overline{\psi}_{E'} \ , \tag{C.102}
$$

$$
\mathcal{H} = -a^{-1}(\pi_a^2 + 36a^2) + 12a^{-1} n^{AA'} \psi_A \overline{\psi}_{A'} \ . \tag{C.103}
$$

Comparing with (5.26), (5.27), (5.28), and (5.29), we see that the redefinition (5.24), (5.25) implies that the last term in (C.101), (C.102), (C.103) is absent. $\Psi \sim \mathsf{A}_1 + \mathsf{A}_2 \psi_A \psi^A$, $\mathsf{A}_1 = \mathrm{e}^{-3a^2}$, and $\mathsf{A}_2 = \mathrm{e}^{3a^2}$ constitute solutions of the equations induced by $\mathcal{S}_A \Psi = 0$ and $\overline{\mathcal{S}}_{A'} \Psi = 0$, respectively. This holds for the pure case if we use either ψ_0^A or ρ^A. This particular Ψ is also a solution of $\mathcal{H}\Psi = 0$, *but only* for the $\mathcal{H}$ *without* the second term in (C.103), i.e., when (5.24), (5.25) are fully employed. In fact, Ψ does not constitute a solution of the full expression in (C.103). The function e^{3a^2} would have to be replaced [27].

5.3 On Conserved Currents

We can derive the conserved current $J^X \sim \Psi^* \nabla^X \Psi - \Psi \nabla^X \Psi^*$ from the Wheeler–DeWitt equation for superspace:

- It satisfies $\nabla_X J^X = 0$, where ∇_X is the corresponding *minisuperspace covariant derivative* (see Sect. 2.8 for notation).
- It associates a flux across a surface Σ with the current J^X. In particular, Σ may be defined as the hypersurface of constant value of the corresponding timelike coordinate in a minisuperspace.
- Moreover, a conserved probability can then be defined from J on the set of classical trajectories. However, this conserved current suffers from difficulties with negative probabilities [6].

The situation corresponding to the last item above bears obvious similarities with the case of a scalar field ϕ satisfying a Klein–Gordon equation. In this case, the surface Σ is usually of constant physical time. But the fact that $J^0[\phi]$ may be negative suggests the need for a Dirac equation. From Dirac's equation, a *new* conserved current was derived, with the advantage of inducing positive-definite probabilities. Subsequently, the concepts of antiparticles and second quantization were introduced. The important point to emphasize here is that the Dirac equation constitutes a square-root of the Klein–Gordon equation. But how far can we stretch this tempting analogy between, on the one hand, the Klein–Gordon and Dirac equations and, on the other hand, the Wheeler–DeWitt and the equations obtained from the supersymmetry (and Lorentz) constraints?

Within a standard quantum cosmological formulation, the possibility for J^0 to be positive or negative may just correspond to having both expanding and collapsing classical universes. The flow will intersect a *generic* Σ at different times. In other words, J^0 being negative is then due to a poor choice of Σ and does not necessarily lead to third quantization. However, the choice of Σ as a surface of constant S within a semiclassical minisuperspace approximation is quite satisfactory. The flow associated with ∇S intersects them once and only once. For more details, see [6, 10, 7].

In the FRW SQC context with a scalar supermultiplet as matter content, let us write down the first-order differential equations derived from the supersymmetry constraints, i.e., (5.74), (5.75), (5.76), (5.77), (5.78), (5.79), and (5.80), but with $\phi \equiv r\mathrm{e}^{i\theta}$, choosing $\mathsf{C} = 0$ (this is important!) and $\mathsf{P} = 0$. This will assist us in getting the explicit dependence of Ψ on ϕ, $\overline{\phi}$ and adequately identify the Hartle–Hawking wave function. We then get

$$\frac{\partial \mathsf{A}}{\partial r} - \mathrm{i}\frac{1}{r}\frac{\partial \mathsf{A}}{\partial \theta} = 0\,, \qquad \frac{\partial \mathsf{E}}{\partial r} + \mathrm{i}\frac{1}{r}\frac{\partial \mathsf{E}}{\partial \theta} = 0\,, \tag{C.104}$$

$$(1+r^2)\frac{\partial \mathsf{B}}{\partial r} - \mathrm{i}\frac{1+r^2}{r}\frac{\partial \mathsf{B}}{\partial \theta} + r\mathsf{B} = 0\,, \qquad (1+r^2)\frac{\partial \mathsf{D}}{\partial r} + \mathrm{i}\frac{1+r^2}{r}\frac{\partial \mathsf{D}}{\partial \theta} + r\mathsf{D} = 0\,. \tag{C.105}$$

After integration, these provide the general quantum state

$$\begin{aligned}\Psi &= c_1 r^{\ell_1}\mathrm{e}^{-\mathrm{i}\ell_1\theta}\mathrm{e}^{-3\sigma^2 a^2} + c_3 a^3 r^{\ell_3}\mathrm{e}^{-\mathrm{i}\ell_3\theta}(1+r^2)^{1/2}\mathrm{e}^{3\sigma^2 a^2}\psi^C\psi_C \\ &\quad + c_4 a^3 r^{\ell_4}\mathrm{e}^{\mathrm{i}\ell_4\theta}(1+r^2)^{1/2}\mathrm{e}^{-3\sigma^2 a^2}\chi^C\chi_C \\ &\quad + c_2 r^{\ell_2}\mathrm{e}^{\mathrm{i}\ell_2\theta}\mathrm{e}^{3\sigma^2 a^2}\psi^C\psi_C\chi^D\chi_D\,,\end{aligned} \tag{C.106}$$

where $\ell_1, \ldots, \ell_4$ and $c_1, \ldots, c_4$ are constants. Notice now the explicit form of A, B, D, and E in (C.106), in contrast with previous expressions. If we had used $\phi = \phi_1 + \mathrm{i}\phi_2$, then the corresponding first-order differential equations would lead to

$$\mathsf{A} = d_1\mathrm{e}^{-3\sigma^2 a^2}\mathrm{e}^{k_1(\phi - \mathrm{i}\phi_2)}\,, \qquad \mathsf{B} = d_3\mathrm{e}^{3\sigma^2 a^2}(1+\phi_1^2+\phi_2^2)\mathrm{e}^{k_3(\phi-\mathrm{i}\phi_2)}\,,$$
$$\mathsf{D} = d_4\mathrm{e}^{-3\sigma^2 a^2}(1+\phi_1^2+\phi_2^2)\mathrm{e}^{k_4(\phi+\mathrm{i}\phi_2)}\,, \qquad \mathsf{E} = d_2\mathrm{e}^{3\sigma^2 a^2}\mathrm{e}^{k_2(\phi+\mathrm{i}\phi_2)}\,.$$

The bosonic coefficients in (C.106) correspond to particular solutions obtainable within the framework of 'purely bosonic' quantum cosmology *if* a specific factor ordering for π_a, π_r, π_θ is used in the Wheeler–DeWitt equation. The point is that the supersymmetry constraints imply that $\partial \mathsf{A}/\partial\phi = 0$ and $\partial \mathsf{E}\partial\overline{\phi} = 0$, and the Wheeler–DeWitt equation involves a term

$$\pi_\phi \pi_{\overline{\phi}} \sim (\pi_r - \mathrm{i}\pi_\theta)(\pi_r + \mathrm{i}\pi_\theta) \longmapsto \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

But

$$\left(\frac{\partial}{\partial r} - \mathrm{i}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial}{\partial r} + \mathrm{i}\frac{1}{r}\frac{\partial}{\partial\theta}\right) \neq \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

Hence, the presence of supersymmetry *selects* a particular factor ordering for the canonical momenta in the Hamiltonian constraint. As a consequence, specific exact solutions (say, $\mathrm{e}^{-3\sigma^2 a^2} r^\ell e^{\mathrm{i}\ell\theta}$) can be found from the Wheeler–DeWitt equation in the gravitational and matter sectors. The no-boundary wave function corresponds to the bosonic coefficient A.

Notice that the bosonic coefficients in (C.106) satisfy interesting relations in a 3D minisuperspace:

$$\frac{\partial(\mathsf{A}\cdot\mathsf{E})}{\partial a} + \frac{\partial(\mathsf{A}\cdot\mathsf{E})}{\partial\theta} - \mathrm{i}r\left(\frac{\partial \mathsf{E}}{\partial r}\mathsf{A} - \frac{\partial \mathsf{A}}{\partial r}\mathsf{E}\right) = 0 , \tag{C.107}$$

$$\check{\partial}_a(\mathsf{B}\cdot\mathsf{D}) + \frac{\partial(\mathsf{B}\cdot\mathsf{D})}{\partial\theta} - \mathrm{i}r\left(\frac{\partial \mathsf{B}}{\partial r}\mathsf{D} - \frac{\partial \mathsf{D}}{\partial r}\mathsf{B}\right) = 0 , \tag{C.108}$$

where $\check{\partial}_a \equiv \partial_a - 6/a$. However, the presence of the last term in both (C.107) and (C.108) clearly prevents us from associating them with a conservation equation of the type $\nabla J = 0$. It is explained in [28] how the presence of supersymmetry, the fact that θ is no longer a cyclical coordinate, and the absence of satisfactory conserved currents, are all related.

5.4 Supersymmetry Invariance for an FRW Model with Vector Fields

Let us start by analysing how the the ansätze for the configuration variables may be affected by a combination of local coordinate, Lorentz, gauge, and supersymmetry transformations. Notice that the tetrad and gravitino are not affected by the action of our internal transformation SU(2) when *no* scalar fields and fermionic partners are present. Use for the tetrad [29]:

$$\begin{aligned}\delta e^{AA'}{}_i =& \left[-\mathcal{N}^{AB} + a^{-1}\xi^{AB} + \mathrm{i}\varepsilon^{(A}\psi^{B)}\right] e_B{}^{A'}{}_i \\ &+ \left[-\overline{\mathcal{N}}^{A'B'} + a^{-1}\overline{\xi}^{A'B'} + \mathrm{i}\overline{\varepsilon}^{(A'}\overline{\psi}^{B')}\right] e^A{}_{B'i} + \frac{\mathrm{i}}{2}\left(\varepsilon_C\psi^C + \overline{\varepsilon}_{C'}\overline{\psi}^{C'}\right) e^{AA'}{}_i ,\end{aligned} \tag{C.109}$$

where ξ^{μ}, $\mathcal{N}^{AB}$, and ε^{A} are time-dependent vectors or spinors parametrizing local coordinate, Lorentz and supersymmetry transformations. A relation like

$$\delta e_i^{AA'} = \mathsf{P}_1 \left[e_{\mu}^{AA'}, \psi_{\mu}^{A} \right] e_i^{AA'}$$

then holds, where P_1 is a spatially independent and possibly complex expression in which all spatial and spinorial indices have been contracted, provided that the relations

$$\mathcal{N}^{AB} - a^{-1}\xi^{AB} - \mathrm{i}\varepsilon^{(A}\psi^{B)} = 0 \, , \qquad \overline{\mathcal{N}}^{A'B'} - a^{-1}\overline{\xi}^{A'B'} - \mathrm{i}\overline{\varepsilon}^{(A'}\overline{\psi}^{B')} = 0 \, , \tag{C.110}$$

between the generators of Lorentz, coordinate, and supersymmetry transformations are satisfied. Hence, we will achieve $\delta e^{AA'}{}_i = \mathsf{P}(t)\delta e^{AA'}{}_i$ with

$$\mathsf{P}(t) \equiv \frac{\mathrm{i}}{2}\left(\varepsilon_C \psi^C + \overline{\varepsilon}_{C'}\overline{\psi}^{C'} \right) \, ,$$

and the ansatz will be transformed into a similar configuration. Notice that any Grassmann-algebra-valued field can be decomposed into a 'body' or component along unity (which takes values in the field of real or complex numbers) and a 'soul' which is nilpotent. The combined variation above implies that $\delta e^{AA'}{}_i$ exists entirely in the nilpotent (soul) part [30, 31].

Under local coordinate, Lorentz, and supersymmetry transformations (when $\phi = \overline{\phi} = 0$), we get

$$\begin{aligned} \delta\psi^{A}{}_i = {} & a^{-1}\overline{\xi}^{A'B'} e^{A}{}_{B'i}\overline{\psi}_{A'} + \frac{3\mathrm{i}}{4}\varepsilon^{A}\psi^{B}\overline{\psi}^{B'} e_{BB'i} \\ & - \frac{3\mathrm{i}}{8}\varepsilon^{A}\lambda^{(a)C} e_{iCC'}\overline{\lambda}^{(a)C'} - \frac{3\mathrm{i}}{4}\varepsilon_C\lambda^{(a)C} e^{A}{}_{C'i}\overline{\lambda}^{(a)C'} \\ & + \left[2\left(\frac{\dot{a}}{a\mathcal{N}} + \frac{\mathrm{i}}{a} \right) - \frac{\mathrm{i}}{2\mathcal{N}}\left(\psi_F\psi_0^F + \overline{\psi}_{F'0}\overline{\psi}^{F'} \right) \right] n_{BA'} e_i^{AA'}\varepsilon^{B} \, . \end{aligned} \tag{C.111}$$

This means that, to recover a relation like $\delta\psi_i^A = \mathsf{P}_2\left[e_{\mu}^{AA'}, \psi_{\mu}^{A} \right] e_i^{AA'}\overline{\psi}_{A'}$, we require $\overline{\xi}^{A'B'} = \xi^{AB} = 0$. In addition, equating the second and third terms in (C.111) with zero gives the contribution of the spin 1/2 ψ and λ fields to the Lorentz constraint. We also need to consider the term $\varepsilon_C\lambda^{(a)C}e^{A}{}_{C'i}\overline{\lambda}^{(a)C'}$ as representing a field variable with indices A and i, for each value of (a). Notice that, to preserve the ansatz, we had to require $n_{AB'}\varepsilon^{B} \sim \overline{\psi}_{B'}$, which is *not* quite $\overline{\psi}$. Here we have to deal with the λ and $\overline{\lambda}$ fields, and a similar step is necessary. Finally, we get

$$\left[2\left(\frac{\dot{a}}{a\mathcal{N}} + \frac{\mathrm{i}}{a} \right) - \frac{\mathrm{i}}{2\mathcal{N}}\left(\psi_F\psi^{F}{}_0 + \overline{\psi}_{F'0}\overline{\psi}^{F'} \right) \right] n_{BA'}\varepsilon^{B} \sim \mathsf{P}_2\left[e_{\mu}^{AA'}, \psi_{\mu}^{A} \right]\overline{\psi}_{A'} \, . \tag{C.112}$$

This means that the variation $\delta\psi_i^A = \mathsf{P}_2(t)\psi_i^A$ with

$$\mathsf{P}_2(t) = \left[2\left(\frac{\dot{a}}{a\mathcal{N}} + \frac{\mathrm{i}}{a}\right) - \frac{\mathrm{i}}{2\mathcal{N}}\left(\psi_F \psi^F{}_0 + \overline{\psi}_{F'0}\overline{\psi}^{F'}\right)\right]$$

will have a component along unity (the body of the Grassmann algebra) and another which is nilpotent (the soul $\psi_F \psi^F{}_0 + \overline{\psi}_{F'0}\overline{\psi}^{F'}$).

Concerning the choice for $A_\mu^{(a)}$, it should be noticed that *only* non-Abelian spin 1 fields can exist consistently within a $k = +1$ FRW background, whose isometry group is SO(4). More specifically, since the physical observables are to be SO(4)-invariant, the fields with gauge degrees of freedom may transform under SO(4) if these transformations can be compensated by a gauge transformation. The idea behind the ansatz is to define a *homomorphism* from the isotropy group SO(3) to the gauge group. This homomorphism defines the gauge transformation which, for the symmetric fields, compensates the action of a given SO(3) rotation. The spin 1 field components in the basis $\left(e^i_{\hat{c}} \mathrm{d}x^{\hat{c}}, \mathcal{T}_{(a)}\right)$ can be expressed as

$$A_i^{(a)} = \frac{f}{2}\delta_i^{(a)} \,. \tag{C.113}$$

The local coordinate and Lorentz transformations will correspond to isometries and local rotations, and these have been compensated by gauge transformations (see [32] for more details). Under SUSY transformations, we get

$$\delta_{(\mathrm{s})} A_i^{(a)} = \mathrm{i} a e_i^b \sigma_{bAA'}\left[\varepsilon^A \overline{\lambda}^{A'(a)} - \lambda^{A(a)}\overline{\varepsilon}^{A'}\right] \,. \tag{C.114}$$

We must therefore impose the condition[10]

$$\begin{cases} \sigma_{bAA'}\left[\varepsilon^A \overline{\lambda}^{A'(a)} - \lambda^{A(a)}\overline{\varepsilon}^{A'}\right] = \mathsf{P}_3(t)\,, & \begin{cases} (a) = b = 1\,, \\ (a) = b = 2\,, \\ (a) = b = 3\,, \end{cases} \\ \sigma_{bAA'}\left[\varepsilon^A \overline{\lambda}^{A'(a)} - \lambda^{A(a)}\overline{\varepsilon}^{A'}\right] = 0\,, & (a) \neq b\,, \end{cases} \tag{C.115}$$

where $\mathsf{P}_3(t)$ is spatially independent and possibly complex, in order to obtain

$$\delta A^i_{(a)} = \mathsf{P}_3\left[e_{AA'\mu}, \psi^A_\mu, A^{(a)}_\mu, \lambda^{(a)}_A\right] A^i_{(a)} \,.$$

It follows from (C.115) that preservation of the ansatz will require $\delta A_i^{(a)}$ to include a nilpotent (soul) component. This consequence is similar to the one for the tetrad.

[10] If we had chosen $\lambda_A^{(a)} = \lambda_A$ for any value of (a), then we would not be able to obtain a consistent relation similar to (C.115), i.e., such that $\delta A_1^{(1)} \sim A_1^{(1)}$ and $\delta A_1^{(2)} \sim A_1^{(2)}$.

As far as the λ fields are concerned, we obtain the following result for a combined local coordinate, Lorentz, supersymmetry, and gauge transformation:

$$\begin{aligned}\delta\lambda_A^{(a)} = &-\frac{1}{2}\mathcal{F}_{0i}^{(a)} e_A^{\;A'i} n_{A'}^{B}\varepsilon_B + \frac{\mathrm{i}}{2}\mathcal{F}_{ij}^{(a)}\varepsilon_{ijk}h^{1/2}n_{AA'}e^{BA'k}\varepsilon_B \\ &-\frac{\mathrm{i}}{4}\psi_{A0}\lambda^{(a)A'}n^{B}{}_{A'}\varepsilon_B - \frac{\mathrm{i}}{8}\psi_0^C n_{CC'}\overline{\lambda}^{(a)C'}\varepsilon_A \\ &-\frac{\mathrm{i}}{4}\overline{\psi}_0^{A'}\lambda_A^{(a)}n^{B}{}_{A'}\varepsilon_B + \frac{\mathrm{i}}{8}\overline{\psi}_{C'0}n^{CC'}\lambda_C^{(a)}\varepsilon_A \\ &-\frac{\mathrm{i}}{4}\overline{\psi}^{A'}n_{AE'}\overline{\lambda}^{(a)E'}n_{A'}^{B}\varepsilon_B - \frac{\mathrm{i}}{16}\overline{\psi}_{E'}\overline{\lambda}^{(a)E'}\varepsilon_A + \kappa^{abc}\zeta^b\lambda_A^{(c)} \\ &-\mathrm{i}\overline{\psi}_{E'}n^{BE'}\overline{\lambda}^{(a)A'}n_{AA'}\varepsilon_B + \mathrm{i}\overline{\psi}_{E'}\overline{\lambda}^{(a)E'}\varepsilon_A - \frac{1}{2}\psi^F\lambda_A^{(a)}\varepsilon_F - \frac{\mathrm{i}}{\sqrt{2}}\psi^C\lambda_C^{(a)}\varepsilon_A \\ &+\frac{\mathrm{i}}{8}\psi_A\lambda_E^{(a)}\varepsilon^A - \frac{\mathrm{i}}{16}\psi_C\lambda^{(a)C}\varepsilon_A + 2\varepsilon_A\lambda_B^{(a)}\psi^B + \varepsilon_C\psi^C\lambda_A^{(a)}\,, \end{aligned} \tag{C.116}$$

where

$$\mathcal{F}_{0i}^{(a)} \equiv \dot{f}\delta_i^{(a)}\,, \qquad \mathcal{F}_{ij}^{(a)} \equiv \frac{1}{4}(2f - f^2)\varepsilon_{ij(a)}\,. \tag{C.117}$$

The last six terms in (C.116) may be put in a more suitable form in order to obtain

$$\delta\lambda_{(a)}^A = \mathsf{P}_4\left[e_{AA'\mu}, \psi_\mu^A, A_\mu^{(a)}, \lambda_A^{(a)}\right]\lambda_{(a)}^A\,.$$

This would require the remaining terms to satisfy a further condition equated to zero.

The above results concerning the transformations of the physical variables are consistent with an FRW geometry *if* the above-mentioned restrictions are satisfied. The attentive reader may be wondering whether these new relations will themselves be invariant under a supersymmetry transformation or even a combination of supersymmetry, Lorentz, and other transformations. Using these relevant transformations, we just obtain *additional* conditions for the previous ones to be invariant. We can proceed with this process in a recursive way, but neither a contradiction nor a clear indication that the relations are invariant is produced.

However, supersymmetry will be one of the features of our model, in spite of the new relations that will be produced. In fact, the constraints of our FRW model satisfy a supersymmetry algebra, i.e., we can obtain $[\mathcal{S}_A, \overline{\mathcal{S}}_B]_D \sim \mathcal{H} + \mathcal{J}_{AB}$, which is fully consistent with supersymmetry. In addition, the solutions obtained by solving the equation $\mathcal{S}_A\Psi = 0$ and its Hermitian conjugate are in agreement with what would be expected if $N = 1$ supergravity is a square-root of gravity. In particular, our solutions are also present in the set of solutions found in [33–35] for the case of a *non*-supersymmetric quantum FRW model with Yang–Mills fields.

5.5 Other Homogeneity Conditions and Quantum States for SUSY Bianchi IX

Using the components for the (Rarita–Schwinger) gravitino field (in an invariant basis) and requiring them to be spatially constant, the ansatz in Sect. 5.2 with the supersymmetry constraints lead to a 'wormhole' (Hawking–Page) state [36], but *not* a 'no-boundary' (Hartle–Hawking) state [37].

In [38], it was shown that the existence of a Hartle–Hawking state depends on whether the spinor components have the same or opposite sign at antipodal points of the spatial 3-manifold. The second choice is necessary if we seek a homogeneous Hartle–Hawking state, since SU(2) (unlike SO(3)) can be continuously shrunk to a point within some smooth 4-manifold.

Before proceeding:

- It is necessary to label points in S^3 with s, as matrices in SU(2).
- Symmetry transformations are represented by $\mathsf{s} \longrightarrow \mathsf{s}u^\dagger$, $\mathsf{u} \in$ SU(2).

Hence, a field f can be said to be homogeneous if it is invariant under the transformations, up to the action of some locally acting group. In other words, if it satisfies

$$\mathsf{f}(\mathsf{s}u^\dagger) = \mathsf{r}\mathsf{f}(\mathsf{s}) \ ,$$

where r is some representation of the matrix $\mathsf{u} \in$ SU(2):

- Note that the simpler and obvious definition constitutes the trivial representation: $\mathsf{r} = 1$ for all $\mathsf{u} \in$ SU(2) (as used in [39]), in which all fields have constant components.
- Another choice is based on the fact that physical fields carry a representation of the group of spatial rotations which are themselves represented by elements of SU(2). This means choosing r to be a rotation represented by the matrix u.
- For example, take a spin 1/2 field χ on $\mathcal{S}^3$. Instead of requiring χ to have constant components when referred to an invariant *dreibein* (i.e., a *dreibein* invariant under the diffeomorphisms $\mathsf{s} \mapsto \mathsf{s}u^\dagger$), we can demand that $\mathsf{s}\chi$ should be constant on $\mathcal{S}^3$.

For the Bianchi IX geometry, let us take the 3-metric in the form $h = h_{ij}\omega^i \otimes \omega^j$, with components $h_{ij} = \mathrm{e}^{2\beta_a}\mathcal{O}_i{}^a\mathcal{O}_j{}^a$ in the invariant basis, where the diagonalising matrix $\mathcal{O}_j^a$ satisfies the orthogonality condition $\delta^{ij}\mathcal{O}_i{}^a\mathcal{O}_j{}^b = \delta^{ab}$, $\det\mathcal{O} = 1$. Note that the three scale parameters β_a represent the physical degrees of freedom of the 3-metric. The three degrees of freedom contained in the matrix $\mathcal{O}$ are pure gauge, and are associated with the group of diffeomorphisms generated by the three Killing vector fields dual to the 1-forms ω^i. By taking the matrix $\mathcal{O}$ to be fixed, we eliminate these unphysical degrees of freedom. We further choose a (preferred) Lorentz frame in which the axes of the dreibein coincide with the main axes of the metric tensor. In this special gauge, the dreibein consists of the three 1-forms

$$e^a \equiv e^a{}_i\omega^i \ , \qquad e^a{}_i \equiv e^{\beta_a}\mathcal{O}_i{}^a \ . \tag{C.118}$$

We then consider the spatial part $\psi_i^A = \psi^{Aa} e_i^a$ of the spin 3/2 Rarita–Schwinger field. Introducing local coordinates x^i on the 3-manifold, with $\omega^i = \omega^i{}_j \mathrm{d}x^j$, we take the homogeneity condition for where the components ψ^{Aa} satisfy

$$\frac{\partial \psi^{Aa}}{\partial x^i} = -\mathrm{i}\omega^p{}_i \mathcal{O}_p{}^b \left(\frac{1}{\sqrt{2}} \sigma_b{}^{AA'} \delta_{BA'} \psi^{Ba} + \mathrm{i}\varepsilon \psi^{Ac} \right) . \tag{C.119}$$

The above means that ψ rotates through an angle 2π relative to an invariant frame as one follows a path between antipodal points on $\mathcal{S}^3$. The components will have opposite signs at antipodal points.

We consider the zero-fermion state, whose wave function Ψ_0 therefore depends only on the 3-geometry. The $\mathcal{S}_A$ constraint is satisfied automatically. The $\overline{\mathcal{S}}_{A'}$ constraint is rewritten as

$$\begin{aligned} -\det[\omega^p{}_i] \left(\frac{1}{2} e_{AA's} \delta^{sr} \psi^A{}_r + \frac{\mathrm{i}\ell}{\sqrt{2}} \varepsilon^{pqr} e_{BA'\,p} \mathcal{O}_q{}^a \sigma_a{}^{BB'} \delta_{AB'} \psi^A{}_r \right) \Psi_0 \\ = \frac{\mathsf{k}^2}{2} \psi^A{}_r \frac{\delta \Psi_0}{\delta e^{AA'}{}_r} , \end{aligned} \tag{C.120}$$

where $e^{AA'}{}_p \omega^p{}_i = e^{AA'}{}_i$, $\psi^A{}_p \omega^p{}_i = \psi^A{}_i$, and the parameter ℓ is 1 for the ansatz (C.119) and vanishes for the 'usual definition' of homogeneity. Cancelling the factors of $\psi^A{}_r$ which appear on both sides, then integrating over the whole of $\mathcal{S}^3$, we obtain

$$\frac{\mathsf{k}^2}{2} \frac{\partial (\ln |\Psi_0|)}{\partial e_r^{AA'}} = (4\pi)^2 \left(\frac{1}{2} e_{AA's} \delta^{rs} + \frac{\mathrm{i}\ell}{\sqrt{2}} \varepsilon^{pqr} e_{BA'p} \mathcal{O}_q{}^a \sigma_a{}^{BB'} \delta_{AB'} \right) . \tag{C.121}$$

Then for (C.118),

$$\Psi_0 \sim \exp\left[-\pi \left(\mathrm{e}^{2\beta_1} + \mathrm{e}^{2\beta_2} + \mathrm{e}^{2\beta_3} - 2\ell \mathrm{e}^{\beta_1+\beta_2} - 2\ell \mathrm{e}^{\beta_2+\beta_3} - 2\ell \mathrm{e}^{\beta_3+\beta_1} \right) \right] . \tag{C.122}$$

For $\ell = 0$, this is the solution first given in [40, 41, 39, 42, 43], i.e., the wormhole state. The choice $\ell = 1$ leads to a Hartle–Hawking state. For a thorough analysis, see [38].

5.6 Quantum States for the SUSY Bianchi IX with a Cosmological Constant

Regarding the $k = +1$ FRW model, a bosonic state was found, namely the Hartle–Hawking solution (see Sect. 5.1.3 for the anti-de Sitter case). But for a diagonal Bianchi IX (a generalized anisotropic extension), the situation has been shown to be far more complicated.

The inclusion of a cosmological constant Λ (see, e.g., [44]) requires the action for $N = 1$ SUGRA to be extended with Υ representing the cosmological constant through the relation $\Lambda \equiv -3\Upsilon^2/2$. The corresponding quantum constraints then read (reinstating $\hbar$)

$$\overline{\mathcal{S}}_{A'}\Psi = -\mathrm{i}\hbar\Upsilon h^{1/2} e^{A}{}_{A'}{}^{i} n_{AB'} D^{BB'}{}_{ji}\left(h^{1/2}\frac{\partial\Psi}{\partial\psi^{B}{}_{j}}\right) + \varepsilon^{ijk} e_{AA'i}\omega^{A}{}_{Bj}\psi^{B}{}_{k}\Psi - \frac{1}{2}\hbar\mathsf{k}^2\psi^{A}{}_{i}\frac{\delta\Psi}{\delta e^{AA'}{}_{i}} = 0\,, \quad \text{(C.123)}$$

$$\mathcal{S}_{A}\Psi = \Upsilon h^{1/2} e^{A}{}_{A'}{}^{i} n_{BA'}\psi_i^B - \mathrm{i}\hbar\omega_{iA}^{B} h^{1/2}\frac{\partial\Psi}{\partial\psi_i^B} - \frac{1}{2}\mathrm{i}\hbar^2\mathsf{k}^2 D_{ji}^{BA'}\left(h^{1/2}\frac{\partial}{\partial\psi^{B}{}_{j}}\right)\frac{\delta\Psi}{\delta e^{AA'}{}_{i}} = 0\,, \quad \text{(C.124)}$$

noticing[11] the replacement $\delta\Psi/\delta\psi^{B}{}_{j} \longrightarrow h^{1/2}\partial\Psi/\partial\psi^{B}{}_{j}$.

Following the procedure in [40], we present this feature in *seven* steps:

1. The gravitino field is written in terms of the β spin 1/2 and γ spin 3/2 modes (see (5.191)), from which we also obtain [26]:

$$\frac{\partial(\beta_A\beta^A)}{\partial\psi^{B}{}_{i}} = -n_{A}{}^{B'} e_{BB'}{}^{i}\beta^{A}\,, \quad \text{(C.125)}$$

$$\frac{\partial(\gamma_{ADC}\gamma^{ADC})}{\partial\psi^{B}{}_{i}} = -2\gamma_{BDC} n^{CC'} e^{D}{}_{C'}{}^{i}\,, \quad \text{(C.126)}$$

$$\frac{\partial\beta_A}{\partial\psi^{B}{}_{j}} = -\frac{1}{2} n_{A}{}^{B'} e_{BB'}{}^{j}\,, \quad \text{(C.127)}$$

$$\frac{\partial\gamma^{ADC}}{\partial\psi^{B}{}_{j}} = \frac{1}{3}\left(n^{CC'} e^{D}{}_{C'}{}^{j}\varepsilon_{B}{}^{A} + n^{AC'} e^{C}{}_{C'}{}^{j}\varepsilon_{B}{}^{D} + n^{DC'} e^{A}{}_{C'}{}^{j}\varepsilon_{B}{}^{C}\right)\,. \quad \text{(C.128)}$$

Moreover, we may write out β^A and γ_{BDC} in terms of $e^{EE'}{}_{j}$ and $\psi^{E}{}_{j}$ to obtain

$$\beta_A = -\frac{1}{2} n_{A}^{A'i} e^{i}_{BA'}\psi_i^B\,, \quad \text{(C.129)}$$

$$\gamma_{ABC} = \frac{1}{3}\left(n_{C}{}^{C'} e_{BC'}{}^{i}\psi_{Ai} + n_{A}{}^{C'} e_{CC'}{}^{i}\psi_{Bi} + n_{B}{}^{C'} e_{AC'}{}^{i}\psi_{Ci}\right)\,. \quad \text{(C.130)}$$

[11] The $h^{1/2}$ factor is needed to ensure that each term has the correct weight in the equations, i.e., when we take a variation of a Bianchi geometry whose spatial sections are compact, multiplying by $\delta/\delta h_{ij}$ and integrating over the three-geometry (see Sect. 5.2.3). It can be checked that inclusion of $h^{1/2}$ gives the correct supersymmetry constraints for a $k = +1$ Friedmann model, when we reduce the diagonal Bianchi IX to a closed FRW (from a direct application of the constraints in $N = 1$ SUGRA).

2. Then, following [40], consider the $\overline{\mathcal{S}}_{A'}\Psi = 0$ constraint at the level ψ^1 in powers of fermions. Similarly to the case of $\Lambda = 0$, since it holds for all $\psi^B{}_i$, we can take the gravitino terms out. Likewise, multiplying this equation by $e^{BA'm}$, and carrying out a variation among the Bianchi IX metrics, we have

$$\hbar \mathsf{k}^2 \frac{\partial \mathsf{A}}{\partial a_1} + 16\pi^2 a_1 \mathsf{A} + 6\pi^2 \hbar \Upsilon a_2 a_3 \mathsf{B} = 0\,, \tag{C.131}$$

and two others given by cyclic permutation of a_1, a_2, a_3. The reader should note the mixing (or term inducing chirality breaking) input by, for example, $16\pi^2 a_1 \mathsf{A} + 6\pi^2 \hbar \Upsilon a_2 a_3 \mathsf{B}$, i.e., where the cosmological constant is present.
3. Next we consider the $\mathcal{S}_A \Psi = 0$ constraint at order ψ^1. Using expressions (C.126) to (C.130), we write β^A and γ^{ABC} in terms of $e^{BD'n}$ and ψ^A_j, then divide again by $\psi^B{}_j$. Multiplying by $n^A{}_{D'} e^{BD'n}$, then multiplying by different choices of $\delta h_{im} = \partial h_{im}/\partial a_1$ and integrating over the manifold, we find the constraints

$$\begin{aligned} 0 = {} & \frac{1}{16}\hbar^2 \mathsf{k}^2 a_1^{-1} \sum_i \left(a_i \frac{\partial \mathsf{B}}{\partial a_i} \right) - \frac{1}{3}\hbar \mathsf{k}^2 \left[3\frac{\partial \mathsf{C}}{\partial a_1} - \mathsf{A}^{-1} \sum_i \left(a_i \frac{\partial \mathsf{C}}{\partial a_i} \right) \right] \\ & -16\pi^2 \Upsilon a_2 a_3 \mathsf{A} \\ & -\pi^2 \hbar a_2 a_3 \left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} \right) \mathsf{B} \\ & +\frac{1}{3}(16\pi^2)\hbar a_2 a_3 \left(\frac{2a_1}{a_2 a_3} - \frac{a_2}{a_3 a_1} - \frac{a_3}{a_1 a_2} \right) \mathsf{C}\,, \end{aligned} \tag{C.132}$$

with two more equations given by cyclic permutation of a_1, a_2, a_3.
4. Now consider the $\overline{\mathcal{S}}_{A'}\Psi = 0$ constraint at order ψ^3. From this constraint we can separately set to zero the coefficient of $\beta^C(\gamma_{DEF}\gamma^{DEF})$, the symmetrized coefficient of $\gamma_{DEF}(\beta_C\beta^C)$, and the symmetrized coefficient of $\gamma_{FGH}(\gamma_{CDE}\gamma^{CDE})$. Three equations can then be derived, following similar steps[12] to [40, 41, 45, 42]:

$$\frac{3}{4}16\pi^2 \hbar \Upsilon a_1 a_2 a_3 \mathsf{D} + \frac{2}{3}\Upsilon 16\pi^2 \left(a_1^2 + a_2^2 + a_3^2 \right) \mathsf{C} + \sum_i \frac{2}{3}\hbar \mathsf{k}^2 \left(a_i \frac{\partial \mathsf{C}}{\partial a_i} \right) = 0 \tag{C.133}$$

and

$$3\hbar \mathsf{k}^2 \frac{\partial \mathsf{B}}{\partial a_1} - \hbar \mathsf{k}^2 \frac{\sum_i a_i \partial \mathsf{B}/\partial a_i}{\mathsf{A}} - 16\pi^2 a_2 a_3 \left(\frac{a_3}{a_1 a_2} + \frac{a_2}{a_1 a_3} - 2\frac{a_1}{a_2 a_3} \right) \mathsf{B} = 0\,, \tag{C.134}$$

and two more equations given by permuting a_1, a_2, a_3 cyclically. Equation (C.134) also holds with B replaced by C.

---

[12] Namely, contracting with $n_C^{A'}$ and integrating out terms of the form $e^{lBA'} n_{CC'} e^n{}_D{}^{C'}$, then using $\delta h_{mn} = \sum_i \partial h_{mn}/\partial a_i$ (see Sect. 5.2.1).

5. Now consider (C.134). It can be checked that these are equivalent to

$$\hbar \mathsf{k}^2 \left(a_1 \frac{\partial \mathsf{B}}{\partial a_1} - a_2 \frac{\partial \mathsf{B}}{\partial a_2} \right) = 16\pi^2 \left(a_2^2 - a_1^2 \right) \mathsf{B} , \tag{C.135}$$

and cyclic permutations. Equation (C.135) can then be integrated, and cyclically, along a characteristic $a_1 a_2 = \text{const.}$, $a_3 = \text{const.}$, say, using the parametric description $a_1 = w_1 \mathrm{e}^{\tau}$, $a_2 = w_2 \mathrm{e}^{-\tau}$, to obtain in the end

$$\mathsf{B} = f(a_1 a_2 a_3) \exp\left[-\frac{8\pi^2}{\hbar \mathsf{k}^2} \left(a_1^2 + a_2^2 + a_3^2 \right) \right] , \tag{C.136}$$

$$\mathsf{C} = \varsigma(a_1 a_2 a_3) \exp\left[-\frac{8\pi^2}{\hbar \mathsf{k}^2} \left(a_1^2 + a_2^2 + a_3^2 \right) \right] . \tag{C.137}$$

6. Substituting these back into (C.132), we get a set of (cyclic) equations (see (C.131)) where we have a term $\Upsilon \mathsf{A}(a_1, a_2, a_3)$ and $(2a_1^2 - a_2^2 - a_3^2)\varsigma$. In order to obtain cyclic symmetry, we must have $\varsigma = 0$, and the only solution is $\mathsf{C} = 0$.
7. Equation (C.132) and its cyclic permutations with $\mathsf{C} = 0$ must be solved consistently with (C.131) and its cyclic permutations. Eliminating A, one finds another equation which implies that $\mathsf{B} = 0$ and subsequently[13] $\mathsf{A} = 0$. Then we can argue, using the duality between $\Psi[e_{AA'i}, \psi_{Bj}]$ and $\overline{\Psi}[e_{AA'i}, \overline{\psi}_{Bj}]$, that $\mathsf{D} = \mathsf{E} = \mathsf{F} = 0$.

Hence, there are *no* physical quantum states obeying the constraint equations in the diagonal Bianchi IX model *if* Ψ has the form given by the ansatz (5.192).

Nevertheless, the improved approach of [46, 47] could not be straightforwardly employed to solve the so-called SQC cosmological constant conundrum. Terms with $\Lambda < 0$ will violate fermionic number conservation in each fermionic sector of Ψ. An *extension* of the ideas present in [46, 47] using Ashtekar variables (see Chap. 7 of Vol. II) has given an insight into how this problem could be solved [48]. Solutions have the form of exponentials of the $N = 1$ supersymmetric Chern–Simons functional.

Perhaps the use of the wider and improved framework in [49] could also be employed. This is a problem yet to be dealt with! Even if there are chiral breaking terms in the supersymmetry constraints, [49] *may still be* of use. In contrast, the approach in [46–48] *cannot* proceed beyond chiral breaking terms, which will not preserve the number of fermions, and lead to mixing of different fermion levels (e.g., when supermatter is present).

[13] The reader should note that, without the presence of terms with $\Upsilon \mathsf{A}$, the quantity A is not restricted in the way described.

Problems of Chap. 6

6.1 Conformal Factor Independence

It follows from (6.10), (6.11), and (6.25) that a superpotential $\mathsf{W}(q)$ is independent of the conformal factor.

6.2 Determining $\Psi_{N=2}$ for the $k = 1$ FRW and Taub Models

For the $k = 1$ FRW model, we have $\mathsf{W} = a^2/2 = \mathrm{e}^{2\alpha}/2$, and for the expansion in Grassmann variables

$$\Psi \equiv \mathsf{A}_+ + \mathsf{A}_-\theta^0 \ , \tag{C.138}$$

the equations to solve from the constraints

$$\begin{aligned} \mathcal{S}_0 &= \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta^0}\left(-a\frac{\partial}{\partial a} + a^2\right) \ , \\ \overline{\mathcal{S}}_0 &= -\mathrm{i}\theta_0\left(a\frac{\partial}{\partial a} + a^2\right) \ , \end{aligned} \tag{C.139}$$

are

$$\begin{aligned} \left(\frac{\partial}{\partial a} - a\right)\mathsf{A}_- &= 0 \ , \\ \left(\frac{\partial}{\partial a} + a\right)\mathsf{A}_+ &= 0 \ , \end{aligned} \tag{C.140}$$

whose solutions are

$$\mathsf{A}_+ = \mathsf{a}_+\mathrm{e}^{-a^2/2} \ , \qquad \mathsf{A}_- = \mathsf{a}_-\mathrm{e}^{a^2/2} \ , \tag{C.141}$$

with $\mathsf{a}_\pm$ constants.

For the Taub (*micro*superspace sector) model, we have $\mathsf{W} = \mathrm{e}^{2\alpha+2x}/3$ (with $x \equiv \beta_+$), and for the expansion in Grassmann variables

$$\Psi \equiv \mathsf{A}_+ + \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1 + \mathsf{C}_2\theta^0\theta^1 \ , \tag{C.142}$$

the equations to solve are

$$
\begin{aligned}
&\left(\frac{\partial}{\partial\alpha}+\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{A}_{+}=0\,,\\
&\left(\frac{\partial}{\partial x}+\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{A}_{+}=0\,,\\
&\left(\frac{\partial}{\partial\alpha}+\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{B}_{1}-\left(\frac{\partial}{\partial x}+\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{B}_{0}=0\,,\\
&\left(-\frac{\partial}{\partial\alpha}+\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{B}_{0}+\left(\frac{\partial}{\partial x}-\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{B}_{1}=0\,,\\
&\left(\frac{\partial}{\partial\alpha}-\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{C}_{2}=0\,,\\
&\left(\frac{\partial}{\partial x}-\frac{2\mathrm{e}^{2\alpha+2x}}{3}\right)\mathsf{C}_{2}=0\,,
\end{aligned}
\tag{C.143}
$$

whose solutions are

$$
\begin{aligned}
&\mathsf{A}_{+}=\mathsf{A}\mathrm{e}^{-R_1^2/3}\,, \qquad \mathsf{C}_2=\mathsf{C}\mathrm{e}^{R_1^2/3}\,,\\
&\mathsf{B}_0=bR_1R_2^2\left(1+\frac{2R_1^2}{9}\right)\,, \qquad \mathsf{B}_1=-bR_1R_2^2\left(1+\frac{2R_1^2}{9}\right)\,,
\end{aligned}
\tag{C.144}
$$

with $R_1=\mathrm{e}^{\alpha+x}$, $R_2=\mathrm{e}^{\alpha-2x}$, A, b, C constants.

6.3 Interpreting the FRW Solution

In the special case of the closed FRW universe, where $\beta_{\pm}=0$, there are only two components: the empty and filled fermionic sectors. It is, however, of relevance, to note that the solution thereby found, if restricted to the FRW case, is *not* the Hartle–Hawking state [37], which would be obtained from A_{-} with $\mathsf{A}_{+}=0$. From Sect. 6.1.1 (see also [50, 51]),

$$
\frac{\mathrm{d}\alpha}{\mathrm{d}\varsigma}=\mathrm{e}^{2\alpha}\,, \qquad \frac{\mathrm{d}\varsigma}{\mathrm{d}t}=\sqrt{\frac{3\pi}{2}}\mathrm{e}^{-3\alpha}\,, \tag{C.145}
$$

which gives the metric

$$
\mathrm{d}s^2=\mathrm{d}t^2+\frac{1}{4}t^2\left[(\omega^1)^2+(\omega^2)^2+(\omega^3)^2\right]\,, \tag{C.146}
$$

describing a tunnelling solution from vanishing scale parameter (at $t=0$) to a given final value of the scale parameter. It also describes the fluctuation from an asymptotically Euclidean 3-metric at $t=-\infty$ to the final value of the scale parameter. The first case may lead to the quantum initiation of an isotropic universe. The second case may again be interpreted as a virtual wormhole.

6.4 No General Conserved Probability Current

A detailed discussion of what follows can be found in [50]. For simplicity we shall consider the case $\mathcal{G}_{XY}\big(= \mathcal{G}^{(0)}_{XY}\big) = \eta_{XY}$ and W real. Furthermore, we shall put $\hbar = 1$. It will be useful to consider, together with the state vector $|\psi\rangle$, its adjoint with respect to the fermionic variables (but not the q-variables), which we denote by $\langle\psi|$. The scalar product $\langle\psi|\psi\rangle$ then involves only a summation over the discrete fermionic components, not an integration over the variables q, i.e., by construction $\langle\psi|\psi\rangle$ is positive and q-dependent.

The supersymmetry constraints are written as

$$(\mathcal{S} + \overline{\mathcal{S}})|\psi\rangle = 0 \,, \\ \mathrm{i}(\mathcal{S} - \overline{\mathcal{S}})|\psi\rangle = 0 \,, \tag{C.147}$$

so let us introduce a fermionic operator ξ^X, χ^Y by

$$\xi^Y \equiv \psi^Y + \overline{\psi}^Y \,, \\ \chi^Y \equiv \mathrm{i}(\psi^Y - \overline{\psi}^Y) \,, \tag{C.148}$$

with the properties

$$\begin{aligned} &\left(\xi^0\right)^\dagger = -\xi^0 \,, \qquad \left(\chi^0\right)^\dagger = -\chi^0 \,, \\ &\left(\xi^i\right)^\dagger = \xi^i \,, \qquad \left(\chi^i\right)^\dagger = -\chi^i \,, \quad i = 1, 2 \,, \\ &\{\xi^Y, \xi^X\} = 2\eta^{YX} = \{\chi^Y, \chi^X\} \,, \\ &\{\xi^Y, \chi^X\} = 0 \,. \end{aligned} \tag{C.149}$$

Multiplying by ξ^0 and χ^0 from the left, we obtain

$$\big(- \partial_0 + \xi^0\xi^j\partial_j + \mathrm{i}\xi^0\chi^Y\mathsf{W}_{|Y}\big)|\psi\rangle = 0 \,, \\ \big(- \partial_0 + \chi^0\chi^j\partial_j - \mathrm{i}\chi^0\xi^Y\mathsf{W}_{|Y}\big)|\psi\rangle = 0 \,, \tag{C.150}$$

and the adjoint equations

$$- \partial_0\langle\psi| - \partial_j\langle\psi|\xi^j\xi^0 + \mathrm{i}\mathsf{W}_{|Y}\langle\psi|(\chi^Y)^+\xi^0 = 0 \,, \\ -\partial_0\langle\psi| - \partial_j\langle\psi|\chi^j\chi^0 - \mathrm{i}\mathsf{W}_{|Y}\langle\psi|(\xi^Y)^+\chi^0 = 0 \,. \tag{C.151}$$

Multiplying the first of the equations above by $\langle\psi|\mathtt{A}$ from the left and by $\mathtt{A}|\psi\rangle$ from the right, where $\mathtt{A}$ is any operator which is independent of the coordinates, and adding the resulting equations, we obtain

$$\begin{aligned} 0 = &-\partial_0\langle\psi|\mathtt{A}|\psi\rangle + \langle\psi|\mathtt{A}\xi^0\xi^j\partial_j|\psi\rangle + (\partial_j\langle\psi|)\xi^0\xi^j\mathtt{A}|\psi\rangle \\ &+\mathrm{i}\mathsf{W}_{|0}\langle\psi|[\mathtt{A}, \xi^0\chi^0]|\psi\rangle + \mathrm{i}\mathsf{W}_{|j}\langle\psi|[\mathtt{A}, \xi^0\chi^j]|\psi\rangle \,. \end{aligned} \tag{C.152}$$

In order to obtain a conservation law, the operator $\mathtt{A}$ must satisfy the conditions

$$\left[\mathtt{A}, \xi^0 \xi^j\right] = 0 = \left[\mathtt{A}, \xi^0 \chi^j\right] = \left\{\mathtt{A}, \xi^0 \chi^0\right\} . \tag{C.153}$$

One conservation law is

$$\partial_0 \langle \psi | \mathtt{A} | \psi \rangle + \partial_j \left(-\langle \psi | \mathtt{A} \xi^0 \xi^j | \psi \rangle \right) = 0 , \tag{C.154}$$

and another is

$$\partial_0 \langle \psi | \mathtt{B} | \psi \rangle + \partial_j \left(-\langle \psi | \mathtt{B} \chi^0 \chi^j | \psi \rangle \right) = 0 , \tag{C.155}$$

with an operator $\mathtt{B}$ independent of the coordinates satisfying

$$\left[\mathtt{B}, \chi^0 \chi^j\right] = \left[\mathtt{B}, \chi^0 \xi^j\right] = 0 = \left\{\mathtt{B}, \chi^0 \xi^0\right\} . \tag{C.156}$$

Unfortunately, among all these conserved currents *there are none* with a positive density $\langle \psi | \mathtt{A} | \psi \rangle$ or $\langle \psi | \mathtt{B} | \psi \rangle$, because

$$\mathrm{Tr}\, \mathtt{A} = -\mathrm{Tr}(\chi^0 \xi^0 \mathtt{A} \xi^0 \chi^0) = -\mathrm{Tr}\, \mathtt{A} , \tag{C.157}$$

and the same equation for $\mathtt{B}$, i.e., the operators $\mathtt{A}$ and $\mathtt{B}$, or any linear combination of them, cannot be positive.

Choosing $\mathtt{A} = \mathtt{B} = 1$, we obtain the balance equations

$$\partial_0 \langle \psi | \psi \rangle + \partial_j (-\langle \psi | \xi^0 \xi^j | \psi \rangle) = \mathrm{i} \mathsf{W}_{|0} \langle \psi | \xi^0 \chi^0 | \psi \rangle \tag{C.158}$$

and

$$\partial_0 \langle \psi | \psi \rangle + \partial_j (-\langle \psi | \chi^0 \chi^j | \psi \rangle) = \mathrm{i} \mathsf{W}_{|0} \langle \psi | \xi^0 \chi^0 | \psi \rangle . \tag{C.159}$$

The α-dependence of W is seen to spoil the conservation laws with a positive probability density. Thus, if $\mathsf{W} = 0$ as in Bianchi type I models, one has conserved currents with a positive density. On the other hand, we find the general transversality condition

$$\partial_j \langle \psi | (\xi^0 \xi^j - \chi^0 \chi^j) | \psi \rangle = 0 . \tag{C.160}$$

6.5 *N*=2 SUSY Bianchi Model with a Gauge Sector

The action is

$$S \sim \int \mathrm{d}^4 x \sqrt{-g} \left(R - \frac{1}{2} F^{(a)}_{\mu\nu} F^{(a)\mu\nu} \right) . \tag{C.161}$$

For simplicity, it will be restricted to a general diagonal Bianchi type axially symmetric spacetime, parametrized by two independent functions of a cosmological time, viz., $b_1(t) = b_2(t)$ and $b_3(t)$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\Big[(\omega^1)^2 + (\omega^2)^2\Big] + b_3^2(t)(\omega^3)^2 , \tag{C.162}$$

where ω^i are basis left-invariant one-forms ($\mathrm{d}\omega^i = C^i_{jk}\omega^j \wedge \omega^k/2$) for the spatially homogeneous three-metrics. Furthermore, we shall use a general ansatz for an SU(2) YM field, compatible with the symmetries of axially symmetric Bianchi type, with two independent real-valued functions $\alpha(t)$ and $\gamma(t)$, viz.,

$$\mathsf{A} \equiv \alpha(t)\left(\omega^1\mathcal{T}_1 + \omega^2\mathcal{T}_2\right) + \gamma(t)\omega^3\mathcal{T}_3 , \tag{C.163}$$

where $\mathcal{T}_i$ are SU(2) group generators, normalized so that $[\mathcal{T}_i, \mathcal{T}_j] = \varepsilon_{ijk}\mathcal{T}_k$.

We then use $\mathcal{G}_{XY}(q)$ as the metric in the extended minisuperspace of spatially homogeneous axially symmetric three-metrics coupled with the corresponding SU(2) Yang–Mills fields, i.e., $q^X = (b_1, b_3, \alpha, \gamma)$, as bosonic components of the superfield. We then introduce the same number of fermionic fields $\overline{\psi}^X$ and ψ^X, and therefore implement an $N = 2$ supersymmetrization, where the potential $V(q)$ satisfies

$$V(q) = \frac{1}{2}\mathcal{G}^{XY}(q)\frac{\partial\mathsf{W}(q)}{\partial q^X}\frac{\partial\mathsf{W}(q)}{\partial q^Y} . \tag{C.164}$$

The kinetic terms for all models have the form

$$-\dot{b}_1^2 b_3 - 2\dot{b}_1\dot{b}_3 b_1 + \dot{\alpha}^2 b_3 + \dot{\gamma}^2\frac{b_1^2}{2b_3} , \tag{C.165}$$

and the only difference between them is due to the potential terms. Using the expression for the metric on the extended 'minisuperspace',

$$\mathcal{G}_{b_1b_1} = -2b_3 , \quad \mathcal{G}_{b_1b_3} = -2b_1 , \quad \mathcal{G}_{\alpha\alpha} = 2b_3 , \quad \mathcal{G}_{\gamma\gamma} = \frac{b_1^2}{b_3} , \tag{C.166}$$

and the explicit form of the potentials, some superpotential solutions of (6.144) are given in Table C.1.

Table C.1 Potentials and superpotentials in the $N = 2$ case with YM fields

	Lagrangian L_0	Superpotential $W = W_{\mathrm{gr}} + \mathbf{W}_{\mathrm{YM}}$
Bianchi I	$-\left[\frac{\mathbf{1}}{\mathbf{b_3}}\boldsymbol{\alpha}^2\boldsymbol{\gamma}^2 + \frac{\mathbf{b_3}}{\mathbf{2b_1^2}}\boldsymbol{\alpha}^4\right]$	$0 + \boldsymbol{\alpha}^2\boldsymbol{\gamma}$
Bianchi IX	$-\left[\frac{\mathbf{1}}{\mathbf{4}}\frac{\mathbf{b_3^3}}{\mathbf{b_1^2}} - \mathbf{b_3} + \frac{\mathbf{1}}{\mathbf{b_3}}\boldsymbol{\alpha}^2(\boldsymbol{\gamma} - \mathbf{1})^2 + \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{b_3}}{\mathbf{b_1^2}}(\boldsymbol{\alpha}^2 - \boldsymbol{\gamma})^2\right]$	$\frac{1}{2}(2b_1^2 + b_3^2) + \left[\boldsymbol{\alpha}^2(\boldsymbol{\gamma} - \mathbf{1}) - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\gamma}^2\right]$
		or
		$\frac{1}{2}(b_3^2 - 4b_1b_3) + \left[\boldsymbol{\alpha}^2(\boldsymbol{\gamma} - \mathbf{1}) - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\gamma}^2\right]$
FRW	$-\frac{3}{2}b\dot{b}^2 + \frac{1}{2}b\dot{\alpha}^2 + \frac{3}{2}b - \frac{1}{2}\frac{(\mathbf{1} - \boldsymbol{\alpha}^2)^2}{\mathbf{b}}$	$\frac{3}{2}b^2 + \left(\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{\alpha}^3 - \boldsymbol{\alpha}\right)$

In order to find possible supersymmetric wave functions in the 1-, 2-, and 3-fermion sectors, one has to investigate the *topology* of the extended minisuperspace. In the Misner–Ryan parametrization of the spacetime metric, viz.,

$$\mathrm{d}s^2 = -\mathcal{N}^2(t)\mathrm{d}t^2 + \frac{1}{6}\mathrm{e}^{2p(t)+2q(t)}\Big[(\omega^1)^2 + (\omega^2)^2\Big] + \frac{1}{6}\mathrm{e}^{2p(t)-4q(t)}(\omega^3)^2 \ , \quad \text{(C.167)}$$

the metric in the extended minisuperspace (C.166) has the simple diagonal form

$$\mathcal{G}_{pp} = -1 \ , \quad \mathcal{G}_{qq} = 1 \ , \quad \mathcal{G}_{\alpha\alpha} = 2\mathrm{e}^{-2p-2q} \ , \quad \mathcal{G}_{\gamma\gamma} = \mathrm{e}^{-2p+4q} \ , \qquad \text{(C.168)}$$

which shows that the *topology* of the extended minisuperspace is equivalent to the Minkowski topology, with all *cohomologies* trivial (see Sect. 4.3 of Vol. II), $H^p\big(\mathcal{M}(\mathcal{G}_{XY})\big) = 0$, $p = 1, 2, 3$, and thus no physical states exist in the 1-, 2-, and 3-fermion sectors since they have zero norm. Similarly, there are *no* physical states except the ones in null and filled fermion sectors in the considered pure gravitational systems.

After quantization, the only nontrivial zero-energy wave functions in null and filled fermion sectors turns out to have a diverging norm, and this fact indicates *spontaneous breaking of supersymmetry*, caused by YM instantons. The spontaneous supersymmetry breaking which takes place if the Yang–Mills field is added to pure gravity is caused in a quasiclassical approach by a YM instanton contribution to the wave function. This contribution provides an energy shift ΔE from a zero level. To estimate this energy shift for EYM systems, an instanton calculation technique can be used, which should also make it possible to find the lowest level normalizable wave function $|\mathsf{f}^1\rangle$, $H|\mathsf{f}_1^{\mathrm{EYM}}\rangle = \Delta E|\mathsf{f}_1^{\mathrm{EYM}}\rangle$ for the relevant models (for more details, see [52]).

6.6 *N*=2 Supersymmetry and Target Space Duality

Following another line of study [53], string theory exhibits a symmetry known as target space duality. In two-dimensional spacetimes, a string cannot tell whether it is propagating on a circle of radius a or of radius a^{-1}. One may therefore transform between theories of radii a and a^{-1} after a suitable translation on the dilaton field. Let us consider this symmetry within the context of the action (6.268) in the absence of loop corrections, i.e., $D = c > 0$. The Hamiltonian is given by

$$\mathcal{H} \sim \mathrm{e}^{-2\phi}\left[\frac{4}{\mathcal{N}^2}\left(\dot{a}\dot{\phi} - a\dot{\phi}^2\right) - ca\right] , \qquad \text{(C.169)}$$

and (C.169) is invariant under the duality transformation

$$a \longrightarrow \frac{1}{a} \ , \qquad \phi \longrightarrow \phi + \ln a \ . \qquad \text{(C.170)}$$

If we introduce the coordinate pair

$$X \equiv a\mathrm{e}^{-\phi} , \qquad Y \equiv \mathrm{e}^{-\phi} , \tag{C.171}$$

the Hamiltonian takes the form

$$\mathcal{H} \sim -\frac{4}{\mathcal{N}^2}\dot{X}\dot{Y} - cXY , \tag{C.172}$$

and invariance under the duality transformation (C.170) is therefore equivalent to an invariance under the simultaneous interchange $X \leftrightarrow Y$.

The momenta conjugate to X and Y are $\mathsf{p}_X = 4\dot{Y}/\mathcal{N}$ and $\mathsf{p}_Y = 4\dot{X}/\mathcal{N}$, respectively. It is convenient to perform a rescaling of these degrees of freedom by defining $\alpha \equiv X^2$ and $\beta \equiv Y^2$. The classical Hamiltonian (C.172) is then given as in Chap. 6, where the non-vanishing components of the minisuperspace metric are $\mathcal{G}_{\alpha\beta} = \mathcal{G}_{\beta\alpha} = (\alpha\beta)^{1/2}$ and the superpotential is $\mathsf{W} = 2c(\alpha\beta)^{1/2}$. There exists a hidden supersymmetry if W satisfies

$$\frac{\partial \mathsf{W}}{\partial \alpha}\frac{\partial \mathsf{W}}{\partial \beta} = c , \tag{C.173}$$

and also respects the duality symmetry $\alpha \leftrightarrow \beta$. One solution satisfying the necessary conditions is $\mathsf{W} = 2\sqrt{c\alpha\beta}$, and since $\{\alpha, \beta\}$ are positive-definite, $c > 0$ is necessary for the solution to be real.

The general supersymmetric wave function may also be found in closed form for this theory. The bosonic functions in the Grassmann basis expansion of the wave function are again given by (6.176), but F has the slightly different form

$$\mathsf{F} = H_n(\eta)H_{n+1}(\xi)\mathrm{e}^{-(\xi^2+\eta^2)/2} , \tag{C.174}$$

where $\xi \equiv c^{1/4}(\alpha^{1/2} + \beta^{1/2})$ and $\eta \equiv c^{1/4}(\alpha^{1/2} - \beta^{1/2})$.

6.7 *N*=2 Supersymmetry in the Dilaton–Gravity Scenario

From the general supersymmetric wave function

$$\Psi = \mathsf{A}_+ + \mathsf{B}_0\theta^0 + \mathsf{B}_1\theta^1 + \mathsf{C}_2\theta^0\theta^1 , \tag{C.175}$$

where the bosonic functions $\mathsf{A}_+, \mathsf{B}_0, \mathsf{B}_1, \mathsf{C}_2$ are functions of α and β alone, the supersymmetry constraints determine a set of coupled, first-order partial differential equations (see [54] for more details):

$$\begin{aligned}
\left(\frac{\partial}{\partial\alpha}+\frac{\partial \mathsf{W}}{\partial\alpha}\right)\mathsf{A}_+ &= 0\,,\\
\left(\frac{\partial}{\partial\beta}+\frac{\partial \mathsf{W}}{\partial\beta}\right)\mathsf{A}_+ &= 0\,,\\
\left(\frac{\partial}{\partial\alpha}+\frac{\partial \mathsf{W}}{\partial\alpha}\right)\mathsf{B}_1-\left(\frac{\partial}{\partial\beta}+\frac{\partial \mathsf{W}}{\partial\beta}\right)\mathsf{B}_0 &= 0\,,\\
\left(\frac{\partial}{\partial\alpha}-\frac{\partial \mathsf{W}}{\partial\alpha}\right)\mathsf{B}_1+\left(\frac{\partial}{\partial\beta}-\frac{\partial \mathsf{W}}{\partial\beta}\right)\mathsf{B}_0 &= 0\,,\\
\left(\frac{\partial}{\partial\alpha}-\frac{\partial \mathsf{W}}{\partial\alpha}\right)\mathsf{C}_2 &= 0\,,\\
\left(\frac{\partial}{\partial\beta}-\frac{\partial \mathsf{W}}{\partial\beta}\right)\mathsf{C}_2 &= 0\,,
\end{aligned} \tag{C.176}$$

whose solution is

$$\begin{aligned}
\mathsf{A}_+ &= \mathrm{e}^{-\mathsf{W}}\,,\\
\mathsf{C}_2 &= \mathrm{e}^{\mathsf{W}}\,,\\
\mathsf{B}_0 &= \frac{\partial \mathsf{F}}{\partial\alpha}+\mathsf{F}\frac{\partial \mathsf{W}}{\partial\alpha}\,,\\
\mathsf{B}_1 &= \frac{\partial \mathsf{F}}{\partial\beta}+\mathsf{F}\frac{\partial \mathsf{W}}{\partial\beta}\,,
\end{aligned} \tag{C.177}$$

where the function $\mathsf{F} = \mathsf{F}(\alpha, \beta)$ is itself a solution to the equation

$$\frac{\partial^2 \mathsf{F}}{\partial\alpha\partial\beta}+\left(\frac{\partial^2 \mathsf{W}}{\partial\alpha\partial\beta}-\frac{\partial \mathsf{W}}{\partial\alpha}\frac{\partial \mathsf{W}}{\partial\beta}\right)\mathsf{F} = 0\,. \tag{C.178}$$

When W is given by $-2\mathrm{i}\sqrt{\alpha\beta}$, (C.178) simplifies to

$$\left(\frac{\partial^2}{\partial w^2}-\frac{\partial^2}{\partial z^2}-w^2+z^2+2\right)\mathsf{F} = 0\,, \tag{C.179}$$

where $\{w, z\}$ are defined as follows. Imposing the additional restriction that β remains positive- or negative-definite for all physically relevant values of the scale factor and dilaton field,[14] we may introduce a third pair of variables defined by

$$w \equiv \sqrt{\gamma\beta}-\sqrt{\alpha}\,, \qquad z \equiv \sqrt{\gamma\beta}+\sqrt{\alpha}\,, \tag{C.180}$$

[14] This restriction on β ensures that the corresponding Wheeler–DeWitt equation will not be of the elliptic type when expressed in terms of w and z.

where $\gamma = \beta/|\beta|$. For real values of α and β, $z \geq |w|$. With

$$\mathcal{N} = \frac{1}{aV}(\alpha\gamma\beta)^{1/2} , \tag{C.181}$$

the action transforms to

$$S \sim -\int \mathrm{d}t \left[\gamma \left(\dot{z}^2 - \dot{w}^2 \right) + \frac{1}{4} \left(z^2 - w^2 \right) \right] . \tag{C.182}$$

This is the action for the constrained oscillator–ghost oscillator system when $\beta < 0$ and the model corresponds to a constrained hyperbolic system if $\beta > 0$. Hence, F may be interpreted physically as the wave function for a quantum system describing two coupled harmonic oscillators that have identical frequencies but a difference in energy of 2. The solution to (C.179) has the separable form

$$\mathsf{F} = H_n \left(\sqrt{\gamma\beta} + \sqrt{\alpha} \right) H_{n+1} \left(\sqrt{\gamma\beta} - \sqrt{\alpha} \right) \mathrm{e}^{-(\alpha+\gamma\beta)} . \tag{C.183}$$

The functions A_+ and C_2 represent the empty and filled fermion sectors of the Hilbert space.

6.8 *N*=2 Supersymmetric LRS Bianchi I Cosmology

From Sect. 6.2.3, we have the following set of coupled, first-order partial differential equations:

$$-\mathrm{i}\frac{\partial \mathsf{A}_+}{\partial \varsigma} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa-\omega\varsigma}\mathsf{A}_+ = 0 , \tag{C.184}$$

$$-\mathrm{i}\frac{\partial \mathsf{A}_+}{\partial u} = 0 , \tag{C.185}$$

$$-\mathrm{i}\frac{\partial \mathsf{A}_+}{\partial v} = 0 , \tag{C.186}$$

$$-\mathrm{i}\frac{\partial \mathsf{B}_1}{\partial \varsigma} + \mathrm{i}\frac{\partial \mathsf{B}_0}{\partial u} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega\varsigma}\mathsf{B}_1 = 0 , \tag{C.187}$$

$$-\mathrm{i}\frac{\partial \mathsf{B}_2}{\partial \varsigma} + \mathrm{i}\frac{\partial \mathsf{B}_0}{\partial v} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega\varsigma}\mathsf{B}_2 = 0 , \tag{C.188}$$

$$-\mathrm{i}\frac{\partial \mathsf{B}_2}{\partial u} + \mathrm{i}\frac{\partial \mathsf{B}_1}{\partial v} = 0 , \tag{C.189}$$

$$-\mathrm{i}\frac{1}{2}\frac{\partial \mathsf{C}^0}{\partial \varsigma} - \mathrm{i}\frac{1}{2}\frac{\partial \mathsf{C}^1}{\partial u} - \mathrm{i}\frac{1}{2}\frac{\partial \mathsf{C}^2}{\partial v} \mp \mathrm{i}\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega\varsigma}\mathsf{C}^0 = 0 . \tag{C.190}$$

From the other constraint, it follows that

$$\mathrm{i}\frac{\partial \mathsf{A}_-}{\partial \varsigma} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\mathsf{A}_- = 0\,, \tag{C.191}$$

$$\mathrm{i}\frac{\partial \mathsf{A}_+}{\partial u} = 0\,, \tag{C.192}$$

$$-\mathrm{i}\frac{\partial \mathsf{A}_-}{\partial v} = 0\,, \tag{C.193}$$

$$-\frac{\mathrm{i}}{2}\frac{\partial \mathsf{C}^1}{\partial \varsigma} - \mathrm{i}\frac{\partial \mathsf{C}^0}{\partial u} \pm \mathrm{i}\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\mathsf{C}^1 = 0\,, \tag{C.194}$$

$$\mathrm{i}\frac{\partial \mathsf{C}^2}{\partial \varsigma} + \mathrm{i}\frac{\partial \mathsf{C}^0}{\partial v} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\mathsf{C}^2 = 0\,, \tag{C.195}$$

$$\mathrm{i}\frac{\partial \mathsf{C}^2}{\partial u} - \mathrm{i}\frac{\partial \mathsf{C}^1}{\partial v} = 0\,, \tag{C.196}$$

$$\mathrm{i}\frac{\partial \mathsf{B}^0}{\partial \varsigma} - \mathrm{i}\frac{\partial \mathsf{B}^1}{\partial u} - \mathrm{i}\frac{\partial \mathsf{B}^2}{\partial v} \mp \mathrm{i}2\sqrt{2\Upsilon}\mathrm{e}^{-\kappa_\omega \varsigma}\mathsf{B}^0 = 0\,. \tag{C.197}$$

Equations (C.184), (C.185), and (C.186) determine that

$$\mathsf{A}_+ = \mathsf{A}_+^0\mathrm{e}^{\mathsf{W}}\,, \tag{C.198}$$

where A_+^0 is an arbitrary constant and

$$\mathsf{W} \equiv \pm\frac{2\sqrt{2\lambda}}{\kappa_\omega}\mathrm{e}^{-\kappa_\omega \varsigma}\,. \tag{C.199}$$

Similarly, from (C.191), (C.192), and (C.193)

$$\mathsf{A}_- = \mathsf{A}_-^0\mathrm{e}^{-\mathsf{W}}\,, \tag{C.200}$$

where A_-^0 is a second, arbitrary constant.

To solve (C.187), (C.188), (C.189), and (C.197), we redefine the functions B_a as follows:

$$\mathsf{B}_a \equiv \hat{\mathsf{B}}_a\mathrm{e}^{\mathsf{W}}\,, \qquad a = (0, 1, 2)\,. \tag{C.201}$$

From the definition of $\mathsf{W}(\varsigma)$ given in (C.199), it follows from (C.187), (C.188), (C.189), (C.197), and (C.201) that

$$\frac{\partial \hat{\mathsf{B}}_1}{\partial \varsigma} - \frac{\partial \hat{\mathsf{B}}_0}{\partial u} = 0\,, \tag{C.202}$$

$$\frac{\partial \hat{\mathsf{B}}_2}{\partial \varsigma} - \frac{\partial \hat{\mathsf{B}}_0}{\partial v} = 0\,, \tag{C.203}$$

$$\frac{\partial \hat{\mathsf{B}}_2}{\partial u} - \frac{\partial \hat{\mathsf{B}}_1}{\partial v} = 0\,, \tag{C.204}$$

$$\frac{\partial \hat{\mathsf{B}}_0}{\partial \varsigma} - \frac{\partial \hat{\mathsf{B}}_1}{\partial u} - \frac{\partial \hat{\mathsf{B}}_2}{\partial v} - 2\kappa_\omega\mathsf{W}\hat{\mathsf{B}}_0 = 0\,. \tag{C.205}$$

Similarly, by introducing the new set of variables $\hat{\mathsf{C}}^b$ defined by

$$\mathsf{C}^b \equiv \hat{\mathsf{C}}^b \mathrm{e}^{-\mathsf{W}} , \tag{C.206}$$

we derive a new set of equations that are equivalent to (C.190), (C.194), (C.195), and (C.196):

$$\frac{\partial \hat{\mathsf{C}}^1}{\partial \varsigma} + \frac{\partial \hat{\mathsf{C}}^0}{\partial u} = 0 , \tag{C.207}$$

$$\frac{\partial \hat{\mathsf{C}}^2}{\partial \varsigma} + \frac{\partial \hat{\mathsf{C}}^0}{\partial v} = 0 , \tag{C.208}$$

$$\frac{\partial \hat{\mathsf{C}}^2}{\partial u} - \frac{\partial \hat{\mathsf{C}}^1}{\partial v} = 0 , \tag{C.209}$$

$$\frac{\partial \hat{\mathsf{C}}^0}{\partial \varsigma} + \frac{\partial \hat{\mathsf{C}}^1}{\partial u} + \frac{\partial \hat{\mathsf{C}}^2}{\partial v} + 2\kappa_\omega \mathsf{W} \hat{\mathsf{C}}^0 = 0 . \tag{C.210}$$

By manipulating (C.202), (C.203), (C.204), and (C.205), we arrive at the following set of *decoupled* equations:

$$\frac{\partial^2 \hat{\mathsf{B}}_2}{\partial \varsigma^2} - 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{B}}_2}{\partial \varsigma} - \frac{\partial^2 \hat{\mathsf{B}}_2}{\partial u^2} - \frac{\partial^2 \hat{\mathsf{B}}_2}{\partial v^2} = 0 , \tag{C.211}$$

$$\frac{\partial^2 \hat{\mathsf{B}}_0}{\partial \varsigma^2} - 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{B}}_0}{\partial \varsigma} + 2{\kappa_\omega}^2 \mathsf{W} \hat{\mathsf{B}}_0 - \frac{\partial^2 \hat{\mathsf{B}}_0}{\partial u^2} - \frac{\partial^2 \hat{\mathsf{B}}_0}{\partial v^2} = 0 , \tag{C.212}$$

$$\frac{\partial^2 \hat{\mathsf{B}}_1}{\partial \varsigma^2} - 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{B}}_1}{\partial \varsigma} - \frac{\partial^2 \hat{\mathsf{B}}_1}{\partial u^2} - \frac{\partial^2 \hat{\mathsf{B}}_1}{\partial v^2} = 0 . \tag{C.213}$$

For example, (C.211) is derived by applying the differential operator $\partial/\partial v$ to (C.205), then acting on (C.203) with $\partial/\partial\varsigma$ and on (C.204) with $\partial/\partial u$. By employing (C.203), we then arrive at (C.211) above. A similar procedure leads to (C.212) and (C.213).

Applying an equivalent technique to (C.207), (C.208), (C.209), and (C.210) results in a set of decoupled equations for the amplitudes $\hat{C}^c$:

$$-\frac{\partial^2 \hat{\mathsf{C}}^2}{\partial \varsigma^2} - 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{C}}^2}{\partial \varsigma} + \frac{\partial^2 \hat{\mathsf{C}}^2}{\partial u^2} + \frac{\partial^2 \hat{\mathsf{C}}^2}{\partial v^2} = 0 , \tag{C.214}$$

$$\frac{\partial^2 \hat{\mathsf{C}}^0}{\partial \varsigma^2} + 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{C}}^0}{\partial \varsigma} - 2{\kappa_\omega}^2 \mathsf{W} \hat{\mathsf{C}}^0 - \frac{\partial^2 \hat{\mathsf{C}}^0}{\partial u^2} - \frac{\partial^2 \hat{\mathsf{C}}^0}{\partial v^2} = 0 , \tag{C.215}$$

$$-\frac{\partial^2 \hat{\mathsf{C}}^1}{\partial \varsigma^2} - 2\kappa_\omega \mathsf{W} \frac{\partial \hat{\mathsf{C}}^1}{\partial \varsigma} + \frac{\partial^2 \hat{\mathsf{C}}^1}{\partial u^2} + \frac{\partial^2 \hat{\mathsf{C}}^1}{\partial v^2} = 0 . \tag{C.216}$$

Equations (C.202), (C.203), (C.204), and (C.205) can be solved analytically if $\hat{\mathsf{B}}_{1,2}$ are independent of ς. Equations (C.211) and (C.213) then imply that these variables satisfy the 2D Laplace equation, subject to the integrability condition (C.204). Equations (C.202) and (C.203) further imply that $\hat{\mathsf{B}}_0$ is independent of $\{u, v\}$, and consistency between (C.205) and (C.212) results in a further integrability constraint, viz.,

$$\frac{\partial \hat{\mathsf{B}}_1}{\partial u} = -\frac{\partial \hat{\mathsf{B}}_2}{\partial v} \,. \tag{C.217}$$

The functional form of B_0 follows immediately on integration of (C.205), namely, $\mathsf{B}_0 = \mathrm{e}^{-\mathsf{W}}$. It is interesting that this is also the wave function (C.200) for the filled fermion sector. Similar conclusions follow for the functions $\hat{\mathsf{C}}^c$. If $\hat{\mathsf{C}}^{1,2}$ are independent of ς, and satisfy the 2D Laplace equation, the integrability condition $\partial \hat{\mathsf{C}}^1/\partial u = -\partial \hat{\mathsf{C}}^2/\partial v$, and (C.209), then the function C^0 is given by the wave function (C.198) for the empty fermion sector, viz., $\mathsf{C}^0 = \mathrm{e}^{\mathsf{W}}$.

6.9 *N*=2 Supersymmetric Kantowski–Sachs Cosmology

It is natural to consider [54] whether the derived Kantowski–Sachs wave function satisfies the Hawking–Page boundary conditions relevant to a 'wormhole' configuration [36]. In fact, classically, a wormhole represents an instanton solution of the Euclidean field equations. At the quantum level, such a state may be interpreted as a solution to the Wheeler–DeWitt equation with boundary conditions such that the wave function should be regular, in the sense that it does not oscillate an infinite number of times when the three-metric degenerates, and that it should be exponentially damped when the three-geometry tends to infinity.

The anisotropic geometry $S^1 \times S^2$ of the Kantowski–Sachs model implies that there are different possible types of wormhole [55]:

- The geometry of the spacetime asymptotes to $\mathbb{R}^3 \times S^1$ if the radius $\tilde{a}_1$ of the circle tends to a constant as the radius of the two-sphere diverges. The wave function is given by $\Psi \propto \exp(-4\tilde{a}_1\tilde{a}_2)$ in the asymptotic limit.
- Alternatively, if the volume of the two-sphere tends to a constant as $\tilde{a}_1 \to \infty$, the geometry is $\mathbb{R}^2 \times S^2$. The corresponding limit for the $\mathbb{R}^2 \times S^2$ wormhole is $\Psi \propto \exp(-\tilde{a}_1^2)$.

After transforming back to the original variables, the bosonic component of the supersymmetric Kantowski–Sachs wave function (6.348) does *not* asymptote to either of these forms. Its interpretation as a quantum wormhole is therefore *not* clear. However, a further solution to the Euclidean Hamilton–Jacobi equation (6.338) that respects the scale factor duality (6.318) of the classical action is given by

$$\mathsf{W} \sim \left(\frac{32}{A^2 - B^2}\right)^{1/2} \mathrm{e}^{(A\varsigma + Bu)/2} \,. \tag{C.218}$$

Consequently, a supersymmetric quantization may be performed with this solution. Due to the non-trivial functional form of (C.218), however, it has not been possible to find analytical solutions for the intermediate fermionic sectors. On the other hand, the empty fermion sector is given by $\Psi \propto \mathrm{e}^{-\mathsf{W}}$, and it is of interest to compare this wave function with the above ground state wormhole wave functions. For example, in the superstring-inspired model, where $\omega = -1$, we find that $\mathsf{W} = 4a_1a_2\mathrm{e}^{-\phi}$. Performing a conformal transformation on the four-metric, viz., $\tilde{g}_{\mu\nu} = \Theta^2 g_{\mu\nu}$, where $\Theta^2 \equiv \mathrm{e}^{-\phi}$, implies that the dilaton field is minimally coupled to gravity in the 'Einstein frame' $\tilde{g}_{\mu\nu}$. In terms of variables defined in this frame, the wave function is given by $\Psi \propto \exp(-4\tilde{a}_1\tilde{a}_2)$, and this is *precisely* the wave function for the $\mathbb{R}^3 \times S^1$ quantum wormhole that arises in the standard Wheeler–DeWitt quantization. This is important because the ground state of the $\mathbb{R}^3 \times S^1$ quantum wormhole has been selected by the supersymmetric quantization procedure.

Moreover, the interior of a Euclidean Schwarzschild black hole has the form of a Kantowski–Sachs metric [56], and it is possible, therefore, that supersymmetric quantum cosmology may relate a black hole interior to a quantum wormhole. It would be interesting to consider this possibility further. For example, such a relationship would have implications for the graceful exit problem of the pre-big bang inflationary scenario [57]. This problem arises because the classical, dilaton-driven inflationary solution becomes singular in a finite proper time. At present, no generally accepted mechanism has been proposed to avoid such a singularity and ensure a smooth transition to the standard, post-big bang expansion. However, an epoch of pre-big bang inflation may be formally interpreted in the Einstein frame in terms of gravitational collapse [58]. If the final state of such a collapse were a non-singular supersymmetric wormhole configuration, such a problem could in principle be avoided. It is intriguing that, whereas the pre- and post-big bang branches are related to one another by the scale factor duality of the classical action, the empty fermion component of the wave function is invariant under such a duality transformation [54].

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